Lecture 21 April 15, 2010

We found that the power radiated by a relativistic particle is given by Liénard,

$$P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec{\beta}}\,)^2 - (\vec{\beta}\times\dot{\vec{\beta}}\,)^2\right].$$

This is an issue for high-energy accelerators. There are two main types, linear and circular.
In a linear accelerator the direction of $\vec{\beta}$ is constant so $\dot{\vec{\beta}} \parallel \vec{\beta}$ and

$$\begin{aligned} P &= \frac{2e^2}{3c}\gamma^6(\dot{\beta})^2 = \frac{2e^2}{3c}\frac{(\dot{\gamma})^2}{\beta^2} = \frac{2e^2}{3m^2c^5}\left(\frac{dE}{dt}\right)^2 \Bigg/ \left(\frac{dx}{cdt}\right)^2 \\ &= \frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2. \end{aligned}$$

So the power radiated is independent of the energy and depends only on the rate of energy change.

The ratio of power lost to power input, dE/dt

$$\frac{P}{dE/dt} = \frac{2e^2}{3m^2c^3}\frac{1}{v}\frac{dE}{dx} \to \frac{2}{3}\frac{r_e}{mc^2}\frac{dE}{dx},$$

where $r_e = e^2/mc^2$ is the *classical radius of the electron*, 2.82 fm. As we are unlikely to increase the energy of an electron by its rest energy in a distance of its tiny classical radius, energy loss in a linac is negligible.

Circular Acclerators

In a synchrotron, particles are in circular orbit with the energy changing slowly but the direction of the momentum changing rapidly, $\dot{\vec{\beta}} = \vec{\omega} \times \vec{\beta} \perp \vec{\beta}$, so

$$\begin{aligned} P &= \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2\right] = \frac{2}{3}\frac{q^2}{c}\gamma^6\omega^2\beta^2\left[1-\beta^2\right] \\ &= \frac{2}{3}\frac{q^2 c}{\rho^2}\gamma^4\beta^4, \end{aligned}$$

where ρ is the orbit radius and we used $c\beta = \omega\rho$.
The energy loss per revolution δE is the integral of this over an orbit, $\delta t = 2\pi/\omega = 2\pi\rho/c\beta$, or

$$\delta E = \frac{4\pi}{3} q^2 \beta^3 \gamma^4 \left[\rho \left\langle \frac{1}{\rho^2} \right\rangle\right]$$

where the expression in braces is given rather than a simple $1/\rho$ in case you design your accelerator to have magnets not completely covering the circumference[1].

[1] See footnote lecture 15 p. 2, slide 4.

Energy loss in synchrotrons

For ultrarelativistic particles, $\beta \to 1$,

$$\delta E \propto E^4/\rho,$$

with the proportionality constant $8.85 \times 10^{-5}\mathrm{m/GeV}^4$ for electrons and $7.80 \times 10^{-18}\mathrm{m/GeV}^4$ for protons.

For Lep, an electron beam of roughly 80 GeV and a radius of about 4 km, the electrons lose nearly a GeV per turn! This is why the ring is as big as it is.

For the LHC, which will have 7 TeV protons at the same radius, I get a loss of only 4 KeV per turn, so energy loss is not the crucial issue for proton synchrotrons, but bending radius is.

Angular Distribution

We derived he complete expression for $F^{\alpha\beta}$ in covariant form

$$F^{\alpha\beta} = \frac{q}{U_\rho(x^\rho - r^\rho(\tau))} \frac{d}{d\tau}\left[\frac{(x-r(\tau))^\alpha U^\beta(\tau) - (x-r(\tau))^\beta U^\alpha(\tau)}{U_\mu(x^\mu - r^\mu(\tau))}\right]\Bigg|_{\tau_0} . \quad (1)$$

but it is often useful to have the expression more explicitly and in three dimensional language. Using $\vec{R}$ as the 3-vector from $r^\alpha(\tau_0)$ to x^α, with magnitude R and direction $\hat{n}$, we have $R^\alpha := x^\alpha - r^\alpha(\tau_0) = (R, R\hat{n})$, $U^\alpha(\tau_0) = (\gamma c, \gamma c\vec{\beta})$

$$\frac{dU^\alpha}{d\tau} = \gamma\frac{dU^\alpha}{dt} = \left(\gamma^4 c\beta\dot{\beta}, c\gamma^2\dot{\vec{\beta}} + c\gamma^4\beta\dot{\beta}\vec{\beta}\right),$$

where we used $\dot{\gamma} = \gamma^3\beta\dot{\beta}$, and we understand that $\vec{\beta}$, $\dot{\vec{\beta}}$ and γ are to be evaluated at the retarded time τ_0.

In (**??**) we then have $d(x^\alpha - r^\alpha(\tau)/d\tau = -U^\alpha$ and $U_\rho(x^\rho - r^\rho(\tau_0)) = R\gamma c(1 - \hat{n}\cdot\vec{\beta})$, but

$$\begin{aligned} \frac{d}{d\tau} U\cdot(x-r) &= -U^2 + (x-r)_\alpha \frac{dU^\alpha}{d\tau} \\ &= -c^2 + R\left(c\gamma^4\beta\dot{\beta} - \hat{n}\cdot\dot{\vec{\beta}}c\gamma^2 - \hat{n}\cdot\vec{\beta}c\gamma^4\beta\dot{\beta}\right). \end{aligned}$$

Thus

$$\begin{aligned} F^{\alpha\beta} &= \frac{q}{R^3\gamma^3c^3(1-\hat{n}\cdot\vec{\beta})^3} \\ &\quad \left[\left(R^\alpha\frac{dU^\beta}{d\tau} - R^\beta\frac{dU^\alpha}{d\tau}\right)Rc\gamma(1-\hat{n}\cdot\vec{\beta})\right. \\ &\quad -(R^\alpha U^\beta - R^\beta U^\alpha) \\ &\quad \left.\left(-c^2 + R\left\{c\gamma^4\beta\dot{\beta} - \hat{n}\cdot\dot{\vec{\beta}}c\gamma^2 - \hat{n}\cdot\vec{\beta}c\gamma^4\beta\dot{\beta}\right\}\right)\right]. \end{aligned}$$

$\vec{E}$

For the electric field, $\vec{E} = F^{i0}\hat{e}_i$ we have

$$\begin{aligned}\vec{E} &= \frac{q}{R^3\gamma^3c^3(1-\hat{n}\cdot\vec{\beta})^3} \\ &\quad \Bigg[\left(R\vec{n}\gamma^4 c\beta\dot{\beta} - R(c\gamma^2\dot{\vec{\beta}} + c\gamma^4\beta\dot{\beta}\vec{\beta})\right)\cdot Rc\gamma(1-\hat{n}\cdot\vec{\beta}) \\ &\qquad -(\gamma c R\vec{n} - R\gamma c\vec{\beta}) \\ &\qquad\quad \left(-c^2 + R\left\{c\gamma^4\beta\dot{\beta} - \hat{n}\cdot\dot{\vec{\beta}}c\gamma^2 - \hat{n}\cdot\vec{\beta}c\gamma^4\beta\dot{\beta}\right\}\right)\Bigg]\end{aligned}$$

Some algebra spelled out in the lecture notes gives

$$\vec{E} = \frac{q(\hat{n}-\vec{\beta})}{R^2\gamma^2(1-\hat{n}\cdot\vec{\beta})^3} + \frac{q}{Rc}\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^3}.$$

$\vec{B}$

For the magnetic field,

$$\begin{aligned}
B_i &= -\frac{1}{2}\epsilon_{ijk}F^{jk} = -\frac{q\epsilon_{ijk}}{R^3\gamma^3c^3(1-\hat{n}\cdot\vec{\beta})^3} \\
&\quad \Big[(Rn_jc\gamma^2\dot{\beta}_k + Rn_jc\gamma^4\beta\dot{\beta}\beta_k)Rc\gamma(1-\hat{n}\cdot\vec{\beta}) \\
&\quad\quad -Rn_j\gamma c\beta_k\left(-c^2+R(c\gamma^4\beta\dot{\beta}(1-\hat{n}\cdot\vec{\beta})-\hat{n}\cdot\dot{\vec{\beta}}c\gamma^2\right)\Big] \\
&= -\frac{q(\hat{n}\times\vec{\beta})_i}{R^2\gamma^2(1-\hat{n}\cdot\vec{\beta})^3} \\
&\quad -\frac{q}{Rc(1-\hat{n}\cdot\vec{\beta})^3}\left[\hat{n}\times\dot{\vec{\beta}}(1-\hat{n}\cdot\vec{\beta})+\hat{n}\times\vec{\beta}\hat{n}\cdot\dot{\vec{\beta}}\right]_i
\end{aligned}$$

so $\vec{B} = \hat{n}\times\vec{E}$. as it should be for a radiation-zone field.

Power Flux

Thus we can derive the expression for the power radiated towards the observer, the flux being given by the Poynting vector

$$\hat{n} \cdot \vec{S} = \frac{c}{4\pi} E^2.$$

At large distances this is

$$\hat{n} \cdot \vec{S}\Big|_{\rm ret} = \frac{q^2}{4\pi c R^2} \left\{ \frac{\hat{n} \times \left[\left(\hat{n} - \vec{\beta}\right) \times \dot{\vec{\beta}}\right]}{\left(1 - \hat{n} \cdot \vec{\beta}\right)^3}\Bigg|_{\rm ret} \right\}^2 .$$

For ultrarelativistic $\beta \approx 1$ particle, near forward direction, $\hat{n} \cdot \vec{\beta} \approx 1$ flux received $\propto \left(1 - \hat{n} \cdot \vec{\beta}\right)^{-6}$.

But the power radiated is only $\propto \left(1 - \hat{n} \cdot \vec{\beta}\right)^{-5}$.

Why?

Energy per Unit Area

The power/unit area $\hat{n} \cdot \vec{S}\Big|_{\text{ret}}$ received during $[t, t + \Delta t]$ is determined by retarded times $[t_e, t_e + \Delta t_e]$ corresponding to emission τ_0's. Light received at $t + \Delta t$ had to travel a distance $\hat{n} \cdot \vec{v} \Delta t_e$ less than the light received at t, so $\Delta t = (1 - \hat{n} \cdot \vec{\beta})\Delta t_e$.

Total energy emitted = total energy received, so power emitted is $(1 - \hat{n} \cdot \vec{\beta})$ times the power received.

Natural to express things in terms of the emission time, t_e, with $t = t_e + R(t_e)/c$

The energy per unit area we receive is

$$\begin{aligned} E/A &= \int dt\, \hat{n}\cdot\vec{S}\Big|_{\text{ret}} = \int dt'\, \hat{n}\cdot\vec{S}\Big|_{t'} \frac{d}{dt'}\left(t' + \frac{R(t')}{c}\right) \\ &= \int dt'\, \hat{n}\cdot\vec{S}\Big|_{t'} \left(1 - \hat{n}\cdot\vec{\beta}\right). \end{aligned}$$

So the expression which determines the energy distribution is

$$\frac{dP}{dA} = \frac{q^2}{4\pi c R^2} \frac{\left(\hat{n}\times\left[\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right)^2}{\left(1-\hat{n}\cdot\vec{\beta}\right)^5}. \qquad (2)$$

Linear acceleration

Let us consider two important special cases. The first has the acceleration in the same direction as the motion, $\dot{\vec{\beta}} \parallel \vec{\beta}$. Then the numerator is $(\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})^2 = \sin^2\theta \dot{v}^2/c^2$, and

$$\frac{dP}{dA} = \frac{q^2\dot{v}^2}{4\pi c^3 R^2}\,\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}.$$

For β close to 1 this is very strongly peeked in the forward direction. The maximum intensity is when

$$\frac{d}{dx}\left(\frac{1-x^2}{(1-\beta x)^5}\right)\Bigg|_{x=\cos\theta} = 0 = \frac{-2x}{(1-\beta x)^5} + \frac{5\beta(1-x^2)}{(1-\beta x)^6}$$

so

$$x = \cos\theta_{\text{max}} = \frac{\sqrt{1+15\beta^2}-1}{3\beta}.$$

With $\beta = \sqrt{1-\gamma^{-2}} \to 1 - 1/(2\gamma^2)$, $x \to 1 - \dfrac{1}{8\gamma^2}$ and $\theta_{\text{max}} \to 1/2\gamma$.

For such small angles, with $\theta << 1$ but without taking $\gamma\theta$ small, the intensity is

$$\frac{dP}{dA} = \frac{8q^2\dot{v}^2}{\pi c^3 R^2}\gamma^8 \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5}.$$

As an example, consider the linear accelerator at SLAC, which accelerates electrons to 50 GeV over a distance of 3 km. At the end, $\gamma_f = 50\mathrm{GeV}/0.511\mathrm{MeV} \approx 10^5$, and, as the travel has been vitually at the speed of light, $\Delta t = 10^{-5}$ s. Assuming the energy gain per meter is constant, $m_e c^2 \dfrac{d\gamma}{dt} = m_e c^2 \gamma^3 \beta\dot{\beta} = m_e c^2 \gamma_f/\Delta t$, so the final value of $\dot{\beta}$ is $1/\gamma_f^2 \Delta t = 10^{-5}/s$. The angle of maximum intensity is $\theta_{\mathrm{max}} = 1/200,000$ rad $= 4.1$ seconds of arc, and the power per sterradian from this single electron at that angle is

$$\frac{2^{11}}{5^5\pi}\frac{e^2\dot{\beta}^2}{c}\gamma^8 = 1.6\times 10^{12}\ \mathrm{W},$$

just from one electron. Why so much?

Total Linac Power

Note this is not the power, it is the power/sterradian. The total power is

$$\begin{aligned} P &\approx 2\pi R^2 \int_0 \theta\, d\theta\, \frac{dP}{dA} = \frac{16 q^2 \dot{v}^2}{c^3}\, \gamma^8 \int_0 \theta\, d\theta\, \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \\ &= \frac{8 q^2 \dot{v}^2}{c^3}\, \gamma^6 \int_0 du \frac{u}{(1+u)^5} = \frac{2q^2}{3c^3}\, \dot{v}^2\, \gamma^6 . \end{aligned}$$

Circular Accelerators

Another important special case is a circular storage ring, where the acceleration is perpendicular to the velocity. Taking $\vec{\beta}$ in the z direction and $\dot{\vec{\beta}}$ in the x, and using the usual spherical angles for $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we may evaluate the numerator of (**??**) as

$$\left(\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right)^2 = \left(\hat{n}\cdot\dot{\vec{\beta}}(\hat{n}-\vec{\beta}) - [\hat{n}\cdot(\hat{n}-\vec{\beta})]\dot{\vec{\beta}}\right)^2$$

$$= (\hat{n}\cdot\dot{\vec{\beta}})^2(\hat{n}-\vec{\beta})^2 - 2(1-\beta\cos\theta)(\hat{n}-\vec{\beta})\cdot\dot{\vec{\beta}}(\hat{n}\cdot\dot{\vec{\beta}}) + (1-\beta\cos\theta)^2(\dot{\vec{\beta}})^2$$

$$= (\hat{n}\cdot\dot{\vec{\beta}})^2(1-2\beta\cos\theta+\beta^2-2(1-\beta\cos\theta)) + (1-\beta\cos\theta)^2(\dot{\vec{\beta}})^2$$

$$= \left[(\sin\theta\cos\phi)^2(-\gamma^{-2}) + (1-\beta\cos\theta)^2\right](\dot{\vec{\beta}})^2$$

so $$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{(\dot{\vec{v}})^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^2}\right].$$

Again this is strongly peaked in the forward direction.

If we take $\theta << 1$, but keeping $\gamma\theta$ to all orders, so $1-\beta\cos\theta \approx (1+\gamma^2\theta^2)/2\gamma^2$, we have

$$\frac{dP}{d\Omega} \approx \frac{2e^2}{\pi c^3}\frac{\gamma^6(\dot{\vec{v}})^2}{(1+\gamma^2\theta^2)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta)^2}\right].$$

The total power radiated in all directions is, from Liénard,

$$P = \frac{2}{3}\frac{e^2}{c^3}(\dot{\vec{v}})^2\gamma^4,$$

as $(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2 = (\dot{\vec{\beta}})^2(1-\vec{\beta}^2) = \gamma^{-2}(\dot{\vec{\beta}})^2$.

But do not be mislead into thinking this is weaker than in the case with $\dot{\vec{\beta}} \parallel \vec{\beta}$, where we had γ^6 instead of γ^4, because it is very hard to accelerate in the direction of β. A 4-force F in the direction of $\vec{\beta}$ produces

$$\frac{d}{d\tau}mc\beta\gamma = F = mc\dot{\beta}(\gamma+\beta^2\gamma^3) = mc\dot{\beta}\gamma^3 \Longrightarrow \dot{\beta} = \frac{F}{mc\gamma^3},$$

while a force in the transverse direction has $mc\gamma\dot{\beta} = F$, or $\dot{\beta} = F/mc\gamma$. So the $(\dot{\vec{\beta}})^2$ is likely to be γ^4 bigger in the transverse case.

In particular, at the LHC, with 7 TeV protons travelling at roughly c around a 4.3 km radius circle have $\dot{\beta} = \omega \times \beta = 1.1 \times 10^4/\text{ s}$, 10^9 times bigger than the electrons at SLAC, even though their γ is a factor of 13 smaller than the γ of the electrons.
Note that for a given size ring, with ultrarelativistic particles travelling at essentially c, the angular velocity and therefore $\dot{\vec{v}}$ is fixed, so the power radiated is proportional to γ^4 or, for a fixed kind of particle, to E^4. This becomes a serious problem at large energies, especially for electrons (as the power radiated is independent of mass for fixed γ).

Frequency of radiation

Consider a particle in ultrarelativistic circular motion, with radius ρ. As its radiation is essentially confined to a direction $\delta\theta = 1/\gamma$, the arc of the circle during which it irradiates a given distant observer is of length $d = \rho/\gamma$, which it does in a time $\delta t = \rho/\gamma v$. This pulse of light has its leading edge travelling towards the observer a distance $D = c\rho/\gamma v$ during this time, while the trailing edge of the pulse is emitted at d, so the pulse has a length $D - d = (\rho/\gamma)(\beta^{-1} - 1) \approx \rho/2\gamma^3$. Thus the duration of the received pulse is $\rho/2c\gamma^3$ which means it contains frequencies up to $\omega_c \sim (c/\rho)\gamma^3$. Thus synchrotrons are a good source of X-rays.