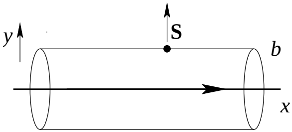
Lecture 20 April 12, 2010

Cherenkov Radiation

First, let's find the energy loss of a heavy fast charged particle differently.

Consider a cylinder of radius b around the track of the projectile. What is flux of energy out of cylinder? The Poynting vector is the flux of escaping energy $\vec{S} = c\vec{E} \times \vec{B}/4\pi$.



We calculated $\vec{E}(0, b, 0)$ earlier, and found $E_z = 0$, so the outward energy flux $S_2 = -cE_1B_3/4\pi$. Integrating the energy flux leaving the cylinder gives the rate of energy loss by the projectile:

$$\frac{\partial E}{\partial t} = \frac{c}{4\pi} 2\pi b \int_{-\infty}^{\infty} dx E_1(x, b, 0, t) B_3(x, b, 0, t).$$

Integrating over x is like integrating over t, so

$$\begin{aligned}\frac{\partial E}{\partial x} = (1/v)\frac{\partial E}{\partial t} &= \frac{c}{4\pi}2\pi b\int_{-\infty}^{\infty} dt\, E_1(0,b,0,t)B_3(0,b,0,t)\\ &= cb\mathrm{Re}\int_0^{\infty} d\omega B_3^*(\omega)E_1(\omega)\end{aligned}$$

where the fields are evaluated at $(0,b,0)$. From last time we have

$$E_1(\omega) = -i\sqrt{\frac{2}{\pi}}\frac{ze\omega}{v^2}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)K_0(\lambda b).$$

We also saw the source for $\vec{A}$ is $\vec{J}$, so it has only an x component, and

$$\begin{aligned}B_3(\vec{k},\omega) &= -ik_2A_1 = -i\epsilon(\omega)k_2(v/c)\Phi(\vec{k},\omega)\\ &= \epsilon(\omega)(v/c)E_2(\vec{k},\omega),\end{aligned}$$

so using the result for E_2 from last time,

$$B_3(\vec{x} = (0,b,0),\omega) = \sqrt{\frac{2}{\pi}}\frac{ze\lambda}{c}K_1(\lambda b).$$

Now we have

$$\left(\frac{dE}{dx}\right) = bc\,\mathrm{Re}\int_0^\infty d\omega\, E_1(\omega)B_3^*(\omega)$$
$$= \frac{2}{\pi}\frac{z^2e^2}{v^2}\mathrm{Re}\int_0^\infty d\omega\,(i\omega\lambda^* b)K_1(\lambda^* b)K_0(\lambda b) \left(\frac{1}{\epsilon(\omega)} - \beta^2\right).$$

This result is due to Fermi.

Cherenkov Radiation

We can use the same calculation to find the flux of energy macroscopically far from the projectile, at a distance a with $\lambda a \gg 1$. We can use the asymptotic forms $K_\nu(z) = \sqrt{\pi/2z}\, e^{-z}$ of the modified Bessel functions, and

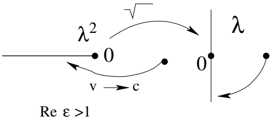
$$\frac{\partial E}{\partial x} == \frac{2}{\pi}\frac{z^2e^2}{v^2}\mathrm{Re}\int_0^\infty d\omega\, \frac{i\omega\lambda^* a}{\sqrt{\lambda\lambda^*}}\frac{\pi}{2a}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right) e^{-2\mathrm{Re}\,\lambda a}.$$

Cherenkov Radiation

Recall

$$\lambda^2 = \frac{\omega^2}{v^2}\left(1 - \beta^2\epsilon(\omega)\right).$$

While ϵ is generally mostly real, it does have a positive imaginary part. For low velocity, or for high ω where $\epsilon \to 1$, λ^2 is basically positive and we are meant to take λ positive, except for a small negative imaginary part. So the energy drops exponentially with distance a. But speed up until $\beta^2 \mathrm{Re}\,\epsilon(\omega) > 1$, then λ becomes imaginary in the lower half plane. $\sqrt{\lambda^*/\lambda} \to i$, and for $|\lambda a| \gg 1$,



$$\left(\frac{dE}{dx}\right) = \frac{z^2e^2}{c^2}\mathrm{Re}\int_{\beta^2\epsilon(\omega)>1} d\omega\,\omega\left(1 - \frac{1}{\beta^2\epsilon(\omega)}\right).$$

Energy not falling off, must be in radiation zone, wave moving in $\vec{E}\times\vec{B}$ direction.

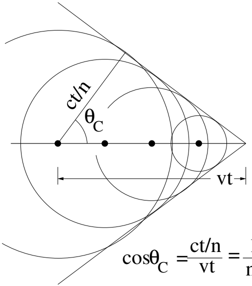
The Cherenkov shock wave

We calculated $\vec{A} \parallel \vec{v}$, so $\vec{B} \perp \vec{v}$, in z direction at $(0, b, 0)$. So direction of wave $\perp \vec{E}$, or $\tan\theta_C = -E_1/E_2$. We found

$$E_1 = i\frac{ze\omega}{c^2}\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right]\frac{e^{-\lambda b}}{\sqrt{\lambda b}}$$

and $E_2 = \dfrac{ze}{v\epsilon(\omega)}\sqrt{\dfrac{\lambda}{b}}e^{-\lambda b}$ so

$$\begin{aligned}\tan\theta_C &= -\frac{E_1}{E_2}\\ &= -i\frac{v\omega\epsilon(\omega)}{c^2\lambda}\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right]\\ &= \sqrt{\beta^2\epsilon(\omega) - 1},\end{aligned}$$



$$\cos\theta_C = \frac{ct/n}{vt} = \frac{1}{n\beta}$$

where I used $\lambda = -i|\lambda|$ in the Cherenkov region. Then

$$\cos\theta_C = \frac{1}{\sqrt{1 + \tan^2\theta_C}} = \frac{1}{\beta\sqrt{\epsilon(\omega)}} = \frac{1}{\beta n(\omega)}.$$

We needed elaborate calculation to find intensity, but every freshman can find θ_C. Consider wavefront from successive circles of emitted light, spreading with speed $c/n = c/\sqrt{\epsilon}$ in the medium. We see right away that $\cos\theta_C = c/nv = 1/\beta n(\omega)$.

Note the polarization is 100% polarized, as $\vec{B}$ is out of the plane.

Hard Scattering

Energy loss from scattering of electrons
Beam direction changed by scattering off heavy particles (nuclei)
Rutherford scattering, dominated by small angles, so

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{2zZe^2}{pv}\right)^2 \frac{1}{\theta^4}$$

with charge of nucleus Ze, p and v of projectile, and θ its scattering angle (in the lab).
Limits of applicability at small and large angles.

Small angles — note $\sigma = 2\pi \int_0 \frac{\sin\theta\, d\theta}{\theta^4} \to \infty$, not right.
We calculated charge of nucleus, ignored screening by electrons in atom — need cutoff for large b.

Fix for small angles

Phenomenological fix for small angles. Take

$$\frac{d\sigma}{d\Omega} = \left(\frac{2zZe^2}{pv}\right)^2 \frac{1}{(\theta^2+\theta_{\text{min}}^2)^2}.$$

θ_{min} not really minimum scattering angle — still have cross section at $\theta = 0$. Several choices, all given by total cross section is roughly πa^2, where a is the radius of electron cloud.

$$\begin{aligned}
\sigma &= 2\pi \left(\frac{2zZe^2}{pv}\right)^2 \int_0^\pi \frac{\sin\theta}{(\theta^2+\theta_{\text{min}}^2)^2} d\theta \\
&\approx 2\pi \left(\frac{2zZe^2}{pv}\right)^2 \int_0^\infty \frac{\theta\, d\theta}{(\theta^2+\theta_{\text{min}}^2)^2} \\
&= \pi \left(\frac{2zZe^2}{pv}\right)^2 \int_0^\infty \frac{du}{(u+\theta_{\text{min}}^2)^2} = \left(\frac{2zZe^2}{pv}\right)^2 \frac{\pi}{\theta_{\text{min}}^2}.
\end{aligned}$$

RMS Scattering Angle

Large angles are not bigger than π. Also, if projectile penetrates nucleus, scattering softens. So set $d\sigma/d\Omega = 0$ for $\theta > \theta_{\max}$.

Projectile suffers many small angle scatterings. Mean change in direction is zero, but average square is

$$
\begin{aligned}
\langle\theta^2\rangle &= \frac{\int \theta^2 \sin\theta (d\sigma/d\Omega)\, d\theta}{\int \sin\theta (d\sigma/d\Omega) d\theta} \approx \frac{\int_0^{\theta_{\max}} \theta^3 d\theta/(\theta^2+\theta_{\min}^2)^2}{\int_0^{\theta_{\max}} \theta d\theta/(\theta^2+\theta_{\min}^2)^2} \\
&= \frac{\int_0^{\theta_{\max}^2} du\, u/(u+\theta_{\min}^2)^2}{\int_0^{\theta_{\max}^2} du/(u+\theta_{\min}^2)^2} \\
&= \frac{\ln(u+\theta_{\min}^2)\big|_0^{\theta_{\max}^2} + \theta_{\min}^2/(\theta_{\max}^2+\theta_{\min}^2) - 1]}{1/\theta_{\min}^2 - 1/(\theta_{\max}^2+\theta_{\min}^2)} \\
&\approx 2\theta_{\min}^2 \ln\frac{\theta_{\max}}{\theta_{\min}}.
\end{aligned}
$$

The number of scatterings in traversing a thickness t is $N\sigma t$, and the mean square of the independent scatterings is the sum of the individual mean squares, so if Θ is the total change in angle (in thickness t),

$$\langle\Theta^2\rangle = N\sigma t\langle\theta^2\rangle = 2\pi N\left(\frac{2zZe^2}{pv}\right)^2 \ln\left(\frac{\theta_{\max}}{\theta_{\min}}\right) t.$$

This fuzziness in the direction of the track will limit the accuracy with which one can determine the initial direction of a charged particle emerging from a collision in a detector, or determine the momentum of a charged particle from its track bending in a magnetic field.

We will skip the rest of Chapter 13.

Radiation by Moving Charges

A charge undergoing a specified motion gives EM radiation. Assuming no incoming field, electromagnetic fields given by the retarded Green's function with the point particle source. Lect. 17

$$D_r(z^\mu) = \frac{\Theta(z^0)}{4\pi R}\delta(z^0 - R), \tag{1}$$

where $R = |\vec{z}|$, and the source of a point particle is

$$J^\mu(x^\nu) = qc\int d\tau\, \delta^4(x^\nu - r^\nu(\tau))U^\mu(\tau), \tag{2}$$

$r^\mu(\tau)$ is world-line (position of charges particle at its proper time τ), $U^\mu(\tau)$ is its 4-velocity.
Note $\Theta(z^0)\delta(z^\mu z_\mu) = \Theta(z^0)\delta(z_0^2 - R^2) = \Theta(z^0)\delta[(z_0 - R)(z_0 + R)] = \frac{1}{2R}\delta(z_0 - R)$, so

$$D_r(z^\mu) = \frac{\Theta(z^0)}{2\pi}\delta(z_\mu z^\mu), \tag{3}$$

The radiation field is thus

$$\begin{aligned} A^\mu(x^\nu) &= \frac{4\pi}{c}\int d^4x' D_r(x-x')J^\mu(x') \\ &= 2q\int d^4x' d\tau \Theta(x^0 - x'^0)\delta((x-x')^2)\delta^4(x^\nu - r^\nu(\tau))U^\mu(\tau) \\ &= 2q\int d\tau \Theta(x^0 - r^0(\tau))\delta((x-r(\tau))^2)U^\mu(\tau). \end{aligned} \tag{4}$$

Use δ function to do $\int d\tau$, using

$$\delta(f(\tau)) = \sum_{\tau_j} \frac{1}{|df/d\tau|_{\tau_j}}\delta(\tau - \tau_j),$$

(where τ_j are the set of points for which $f(\tau)$ vanishes). Here that means $r^\mu(\tau)$ lies on the light-cone of x^μ, and the Θ restricts us to the *backward* light cone. So have only one point, in the past, when the particle crossed the light cone.

As $d(x - r(\tau))^2/d\tau = -2(x^\rho - r^\rho(\tau))U_\rho(\tau)$, we find

$$A^\mu(x^\nu) = q\, \frac{U^\mu(\tau)}{(x^\rho - r^\rho(\tau))U_\rho}\bigg|_{\tau_0},$$

where τ_0 is the point of crossing the light cone.
This is the Liénard-Wiechert potential.

$F^{\mu\nu}$

To get $\vec{E}$ and $\vec{B}$, or $F^{\mu\nu}$, differentiate:

$$\begin{aligned}\partial^\alpha A^\beta &= 2q\int d\tau\Big[\left(\partial^\alpha\Theta(x^0-r^0(\tau))\right)\delta((x-r(\tau))^2)U^\mu(\tau)\\ &\qquad +\Theta(x^0-r^0(\tau))\partial^\alpha\delta((x-r(\tau))^2)U^\mu(\tau)\Big].\end{aligned}$$

In the first term, $\partial^\alpha\Theta(x^0-r^0(\tau))=\delta^\alpha_0\delta(x^0-r^0(\tau))$, contributes only if x^μ and $r^\mu(\tau)$ are at the same time, but the δ function requires $r^\mu(\tau)$ is on the light-cone of x^μ, so it is zero unless x^μ is on the path of the particle, which we will ignore. What remains contains $\partial^\alpha\delta(f(x^\mu,\tau))$, where $f=(x^\mu-r^\mu(\tau))^2$.

As the delta function only depends on f, the chain rule says

$$\begin{aligned}\partial^\alpha \delta(f(x^\mu,\tau)) &= \left(\frac{d}{df}\delta(f)\right)\partial^\alpha f \\ &= 2(x^\alpha - r^\alpha(\tau))\left(\frac{df}{d\tau}\right)^{-1}\frac{d}{d\tau}\delta(f) \\ &= -\frac{(x-r(\tau))^\alpha}{(x-r(\tau))_\rho U^\rho}\frac{d}{d\tau}\delta(f).\end{aligned}$$

Then, plugging in and integrating by parts,

$$\begin{aligned}\partial^\alpha A^\beta &= -2q\int d\tau \Theta(x^0 - r^0(\tau))U^\mu(\tau)\frac{(x-r(\tau))^\alpha}{(x-r(\tau))_\rho u^\rho} \\ &\qquad \frac{d}{d\tau}\delta((x^\mu - r^\mu(\tau))^2) \\ &= 2q\int d\tau \theta(x^0 - r^0(\tau))\delta((x^\mu - r^\mu(\tau))^2) \\ &\qquad \frac{d}{d\tau}\left(\frac{u^\mu(\tau)(x-r(\tau))^\alpha}{(x-r(\tau))_\rho u^\rho}\right),\end{aligned}$$

We have again ignored the $d\Theta/d\tau$ term and we have discarded surface terms. The $\int d\tau \delta(x^\mu - r^\mu(\tau)$ gives a $U_\beta(x^\beta - r^\beta)$ in the denominator, so

$$F^{\alpha\beta} = \frac{q}{U_\rho(x^\rho - r^\rho(\tau))} \frac{d}{d\tau}\left[\frac{(x-r(\tau))^\alpha U^\beta(\tau) - (x-r(\tau))^\beta U^\alpha(\tau)}{U_\mu(x^\mu - r^\mu(\tau))}\right]\Bigg|_{\tau_0}. \tag{5}$$

Discussing this expression

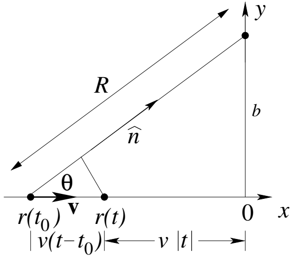
The τ derivative either acts on a U^α, giving an acceleration, or on an r^α. The expression in [ ] is unsuppressed far from the path, so overall F could fall like $1/r$, but when the derivative acts on an r^α, it either kills a power in the numerator or adds one in the denominator, so these terms fall off more rapidly.

Uniformly Moving Charge

Suppose $\vec{v}$ is constant, so is U^α, and the derivative acts on one $x^\sigma - r^\sigma(\tau)$ giving $-U^\sigma$. The terms from differentiating the numerator cancel, so we get

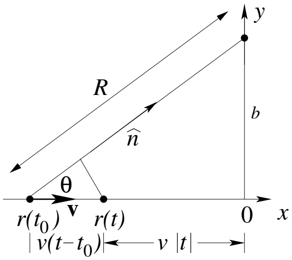
$$F^{\alpha\beta} = qc^2 \frac{(x - r(\tau))^\alpha U^\beta(\tau) - (x - r(\tau))^\beta U^\alpha(\tau)}{(U_\rho(x^\rho - r^\rho(\tau)))^3}.$$

Take $\vec{v}$ along x axis, with $r_x = vt$, and let's observe from $(0, b, 0)$, so $U^\alpha = (\gamma c, \gamma v, 0, 0)$, $r^\alpha(\tau) = U^\alpha \tau$, $x^\mu = (ct, 0, b, 0)$. The particle left the light-cone at time t_0 for which $(x^\mu - r^\mu(t_0))^2 = 0$. $x^\mu - r^\mu(t_0) = (c(t - t_0), -vt_0, b, 0)$, so $c^2(t-t_0)^2 - v^2 t_0^2 - b^2 = 0$. $t_0 = \gamma^2(t - \sqrt{t^2\beta^2 + b^2/c^2\gamma^2})$.



Its Electric Field

The diagram shows where the particle was at t_0, when it exitted our lightcone, and where it is now at time t (which is < 0). In F's denominator,



$$
\begin{aligned}
&U_\alpha(x^\alpha - r^\alpha(t_0)) \\
&\quad = \gamma(c^2(t-t_0) + v^2 t_0) \\
&\quad = c^2\gamma(t - \gamma^{-2}t_0) \\
&\quad = c^2\gamma\sqrt{t^2\beta^2 + b^2/c^2\gamma^2} = c\sqrt{b^2 + v^2\gamma^2 t^2}.
\end{aligned}
$$

Let us evaluate the y component of the electric field:

$$E_2 = F_{02} = qc^2 \frac{(x-r)_2 U_0}{(U_\alpha (x-r)^\alpha)^3} = \frac{qb\gamma}{(b^2 + v^2\gamma^2 t^2)^{3/2}}.$$

For nonrelativistic speeds, $b^2 + v^2\gamma^2 t^2 \to R^2$, so $E_2 \to q\frac{b}{R^3} = q\left(\frac{\vec{x}-\vec{r}}{|\vec{x}-\vec{r}|^3}\right|_y$ as Coulomb told us. But relativistically, the field is squeezed in the direction of the motion.

Power Radiated

We just considered a non-accelerating charge, and we could have found these results by Lorentz transforming the Coulomb field of a particle at rest.

Now consider an accelerating particle and the power it radiates. The Poynting vector gives the flux

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} \to \frac{c}{4\pi}\vec{E}^{\,2}\hat{n}.$$

General power distribution requires evaluating from (5), but instantaneous power is invariant as energy and time transform the same way, so let's calculate it in the particle's instantaneous rest frame. $\dfrac{dU^\alpha}{d\tau} = (0, \dot{\vec{v}})$,

$$r^\mu = (ct - R, \vec{0}),\ x^\mu - r^\mu = (R, \vec{R}) = R(1, \hat{n}),\ U_\alpha(x-r)^\alpha = Rc.$$

In calculating $E_i = F_{0i}$ from (5), the derivative of the numerator

$$\frac{d}{d\tau}\left[(x - r(\tau))^0 U^i(\tau) - (x - r(\tau))^i U^0(\tau)\right]\Bigg|_{\tau_0} = R\,(-\dot{\vec{v}}) - \vec{r}\cdot 0$$

while the derivative of the denominator is $\vec{R}\cdot\dot{\vec{v}}$. Thus

$$\begin{aligned}\vec{E} &= \sum_i F_{0i}\hat{e}_i = \frac{q}{Rc}\left[\frac{R(-\dot{\vec{v}})}{cR} - \frac{-c\vec{R}\,(-\dot{\vec{v}})\cdot\vec{R}}{c^2R^2}\right] \\ &= -\frac{q}{c^2R}\left[\dot{\vec{v}} + \hat{n}\,\hat{n}\cdot\dot{\vec{v}}\right] \\ &= \frac{q}{c^2R}\hat{n}\times(\hat{n}\times\dot{\vec{v}}).\end{aligned}$$

Then the power per sterradian is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\hat{n}\times\dot{\vec{v}}|^2 = \frac{q^2}{4\pi c^3}|\dot{\vec{v}}|^2\sin^2(\psi),$$

where ψ is the angle between the acceleration and the vector $\hat{n}$ pointing to the observer. The integral gives

$$P = \frac{2q^2}{3c^3}|\dot{\vec{v}}|^2.$$

This is the power radiated in the momentary rest frame.

Power in any frame

Jackson argues that we can get the relativistic equation be noting that the power needs to be an invariant expression built from U^α (or p^α) and the first derivative $dp^\alpha/d\tau$. The formula in the rest frame can be expressed as

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt} = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\alpha}{d\tau}\frac{dp_\alpha}{d\tau} \qquad \text{in the rest frame,}$$

but the last expression is invariant. In any other frame, it gives

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right].$$

As $E = mc^2\gamma$, $\vec{p} = mc\gamma\vec{\beta}$, and $d/d\tau = \gamma d/dt$, noting from $\gamma^{-2} = 1-\beta^2$ that $-2\gamma^{-3}d\gamma = -2\beta d\beta$, so $d\gamma = \gamma^3\beta d\beta$, the term in brackets is

$$m^2c^2\gamma^2\left[\left(\gamma^3\beta\dot{\beta}\vec{\beta} + \gamma\dot{\vec{\beta}}\right)^2 - (\gamma^3\beta\dot{\beta})^2\right]$$

,

Then

$$
\begin{aligned}
P &= \frac{2q^2}{3c}\gamma^2 \left[\left(\gamma^3 \beta \dot{\beta} \vec{\beta} + \gamma \dot{\vec{\beta}} \right)^2 - (\gamma^3 \beta \dot{\beta})^2 \right] \\
&= \frac{2q^2}{3c}\gamma^2 \left[\gamma^6 \beta^4 (\dot{\beta})^2 + 2\gamma^4 \beta \dot{\beta} \vec{\beta} \cdot \dot{\vec{\beta}} + \gamma^2 (\dot{\vec{\beta}})^2 - \gamma^6 \beta^2 \dot{\beta}^2 \right] \\
&= \frac{2q^2}{3c} \left[\gamma^6 \dot{\beta}^2 \left(\gamma^2 \beta^4 - \gamma^2 \beta^2 + 2\beta^2) \right) + \gamma^4 (\dot{\vec{\beta}})^2 \right]
\end{aligned}
$$

because $\vec{\beta} \cdot \dot{\vec{\beta}} = \frac{1}{2} d\vec{\beta}^2/dt = \frac{1}{2} d\beta^2/dt = \beta\dot{\beta}$. But $\gamma^2(\beta^4 - \beta^2) = -\beta^2$, so

$$
P = \frac{2}{3}\frac{q^2}{c}\gamma^6 \left(\gamma^{-2} (\dot{\vec{\beta}})^2 - \beta^2 \dot{\beta}^2 \right).
$$

The parentheses may be rewritten $(\dot{\vec{\beta}})^2 - \beta^2 \left((\dot{\vec{\beta}})^2 - \dot{\beta}^2 \right) = (\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2$ because $(\vec{\beta} \times \dot{\vec{\beta}})^2 = (\vec{\beta})^2 (\dot{\vec{\beta}}) - (\vec{\beta} \cdot \dot{\vec{\beta}})^2$ and the last term is $-\beta^2\dot{\beta}^2$ as explained above. So all in all,

$$
P = \frac{2}{3}\frac{q^2}{c}\gamma^6 \left[(\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right].
$$

A reading assignment

The rest of section 14.2 is certainly important but straightforward, so I will not rewrite it. You should read it.