

# Lecture 19 April 8, 2010

## Charged particle Energy Loss

Fast heavy ( $\gg m_e$ ) charged particle interacting with matter.  
 Collisions with electrons transfers lots of energy, not much deviation of particle.  
 Collisions with nuclei, if  $m \ll m_{\text{Nuc}}$ , scattering important but not much energy loss.

Consider first the scattering with electrons, of a projectile with  $v$ ,  $M$ , and  $q = ze$ , with  $E = M\gamma c^2$ ,  $P = M\beta\gamma c$ . Ignore binding of electron in atom, and its initial velocity. It has mass  $m$  and charge  $-e$ .  $M \gg m$ , so in projectile's rest frame, electron Coulomb scatters, with the "well-known Rutherford scattering" cross section<sup>1</sup>.

<sup>1</sup>See apology in lecture notes

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$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2vp}\right)^2 \frac{1}{\sin^4(\theta/2)}, \quad (1)$$

where  $p = m\beta\gamma c$  is the momentum of the electron.  
 We want the change of the projectile's momentum, so need cross section in terms of momentum transfer rather than angle:  
 $Q^2 = -(p'^\mu - p^\mu)^2 > 0$ , which for elastic scattering will be

$$Q^2 = 4p^2 \sin^2(\theta/2), \quad dQ^2 = 2p^2 \sin \theta d\theta,$$

so

$$d\Omega = 2\pi \sin \theta d\theta = \frac{\pi}{p^2} dQ^2, \quad \frac{d\sigma}{dQ^2} = \frac{\pi}{p^2} \frac{d\sigma}{d\Omega} = 4\pi \left(\frac{ze^2}{vQ^2}\right)^2.$$

Note I didn't Lorentz transform the cross section, because  $d\sigma$  is an area transverse to the relative velocity back to the lab frame.

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In projectile's frame,  $P^\mu = (Mc, \vec{0})$ ,  $p^\mu = (mc\gamma, -m\gamma\vec{v})$ , so  $P \cdot p = Mmc^2\gamma$ , so  $\beta^2 = (Mmc^2/P \cdot p)^2$ . Energy lost to electron is  $T = (p'^0 - p^0)c$  in the lab, where  $p^\mu = (mc, \vec{0})$ , so  $mT = p \cdot (p' - p) = p \cdot p' - p^2 = -\frac{1}{2}(p' - p)^2 = \frac{1}{2}Q^2$ . Replacing  $Q^2$  by  $2mT$  on both sides of the cross section equation,

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 e^4}{mv^2 T^2}.$$

This formula will tell us how rapidly a swift projectile loses energy, but its validity is limited:

1)  $T = \frac{Q^2}{2m} = 2\frac{v^2}{m} \sin^2\left(\frac{\theta}{2}\right) \leq 2m(c\beta\gamma)^2$ , so the cross section for  $T > T_{\text{max}} := 2m(c\beta\gamma)^2$  is zero. 2) Lower bound: Unless the projectile gives up enough energy to free the electron from the atom (or at least raise it to a higher quantum state), no energy will be lost, and the cross section should be zero. Call this energy  $\epsilon$ .

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## Energy loss $dE/dx$

Material has  $N$  atoms/unit volume,  $Z$  electrons/atom. Projectile loses energy  $T \pm dT/2$  for each of  $ZNdxd\sigma/dT$  electrons it scatters off with that  $dT$ , so

$$\begin{aligned} -\frac{dE}{dx} &= NZ \int_{\epsilon}^{T_{\text{max}}} T \frac{d\sigma}{dT} dT = 2\pi NZ \frac{z^2 e^4}{mv^2} \int_{\epsilon}^{T_{\text{max}}} \frac{1}{T} dT \\ &= 2\pi NZ \frac{z^2 e^4}{mv^2} \ln\left(\frac{T_{\text{max}}}{\epsilon}\right) \\ &= 2\pi NZ \frac{z^2 e^4}{mv^2} \ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right). \end{aligned}$$

Lots of corrections to this:

$$\text{Dirac spin } \ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right) \rightarrow \ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right) - \beta^2.$$

Energy loss  $< \epsilon$  is not negligible, in fact it doubles  $dE/dx$ .

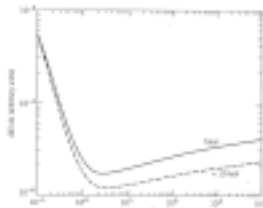
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## Features of $dE/dx$

But basic features given correctly:

- For  $\beta \ll 1$ , proportional to  $1/v^2$ , with coefficient  $\propto NZ$  or the material's density. So loss per gram/cm<sup>2</sup> is roughly material-independent.
- For  $\beta \sim 1$ , grows logarithmically. Therefore a minimum ionizing value, somewhere around  $\beta\gamma = 3$ .



$dE/dx$  from Jackson

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## Coherent Effects

[Note: we are skipping section 2.]

For  $b \gg a \approx N^{-1/3}$  interaction with polarizable medium more appropriate than incoherent scattering by individual atoms.

Assume  $\epsilon(\omega)$  but  $\mu = 1$  for material. Maxwell's laws (in Gaussian units) become:

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad (2)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (3)$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c} \vec{J} \quad (4)$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad (5)$$

with

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}.$$

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## Fourier transforming

Fourier transform everything:

$$F(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d^3k \int d\omega F(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t},$$

we get

$$\begin{aligned} \vec{E}(\vec{k}, \omega) &= -i\vec{k}\Phi(\vec{k}, \omega) + \frac{i\omega}{c}\vec{A}(\vec{k}, \omega) \\ \vec{D}(\vec{k}, \omega) &= -i\epsilon(\omega)\vec{k}\Phi(\vec{k}, \omega) + \frac{i\omega\epsilon(\omega)}{c}\vec{A}(\vec{k}, \omega) \\ \vec{B}(\vec{k}, \omega) &= i\vec{k} \times \vec{A}(\vec{k}, \omega) \end{aligned}$$

so (2) and (4) become

$$\begin{aligned} \epsilon(\omega)k^2\Phi(\vec{k}, \omega) - \frac{\omega\epsilon(\omega)}{c}\vec{k} \cdot \vec{A}(\vec{k}, \omega) &= 4\pi\rho(\vec{k}, \omega) \\ -\vec{k} \times (\vec{k} \times \vec{A}(\vec{k}, \omega)) + \frac{\omega}{c}\epsilon(\omega)\vec{k}\Phi(\vec{k}, \omega) \\ - \frac{\omega^2\epsilon(\omega)}{c^2}\vec{A}(\vec{k}, \omega) &= \frac{4\pi}{c}\vec{J}(\vec{k}, \omega) \end{aligned}$$

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Gauge invariance:  $A^\mu \rightarrow A^\mu - \partial^\mu\Lambda$ , or  $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$ ,  
 $\Phi \rightarrow \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$ , is OK even in materials, so choose *modified*  
Lorenz condition

$$\frac{\epsilon}{c}\frac{\partial\Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0, \quad \text{or } \vec{k} \cdot \vec{A}(\vec{k}, \omega) = \frac{\omega\epsilon(\omega)}{c}\Phi.$$

Then we can write our equations as

$$\begin{aligned} \epsilon(\omega)k^2\Phi(\vec{k}, \omega) - \frac{\omega^2\epsilon^2(\omega)}{c^2}\Phi(\vec{k}, \omega) &= 4\pi\rho(\vec{k}, \omega) \\ k^2\vec{A}(\vec{k}, \omega) - \frac{\omega^2\epsilon(\omega)}{c^2}\vec{A}(\vec{k}, \omega) &= \frac{4\pi}{c}\vec{J}(\vec{k}, \omega) \end{aligned}$$

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## Interaction at large $b$

Projectile with  $\vec{v}$ , will be essentially unchanged (or only slowly diminished)

$$\rho(\vec{x}, t) = ze\delta^3(\vec{x} - \vec{v}t), \quad \vec{J}(\vec{x}, t) = \vec{v}\rho(\vec{x}, t) = ze\vec{v}\delta^3(\vec{x} - \vec{v}t),$$

which means the fourier transformed source is

$$\begin{aligned} \rho(\vec{k}, \omega) &= \frac{ze}{(2\pi)^2} \int d^3x dt \delta^3(\vec{x} - \vec{v}t) e^{-i\vec{k}\cdot\vec{x} + i\omega t} \\ &= \frac{ze}{(2\pi)^2} \int dt e^{-i(\vec{k}\cdot\vec{v} - \omega)t} = \frac{ze}{2\pi} \delta(\omega - \vec{k} \cdot \vec{v}) \end{aligned}$$

and  $\vec{J}(\vec{k}, \omega) = \vec{v}\rho(\vec{k}, \omega)$ . In Fourier space the equations for  $\Phi$  and  $\vec{A}$  become trivial to solve:

$$\begin{aligned} \Phi(\vec{k}, \omega) &= \frac{2ze}{\epsilon(\omega)} \frac{\delta(\omega - \vec{k} \cdot \vec{v})}{k^2 - \omega^2\epsilon(\omega)/c^2}, \quad \vec{A}(\vec{k}, \omega) = \frac{\vec{v}\epsilon(\omega)}{c}\Phi(\vec{k}, \omega) \\ \vec{E}(\vec{k}, \omega) &= -i\vec{k}\Phi(\vec{k}, \omega) + i\frac{\omega}{c}\vec{A}(\vec{k}, \omega) \\ &= \left( -i\vec{k} + i\frac{\omega\epsilon(\omega)}{c^2}\vec{v} \right) \Phi(\vec{k}, \omega). \end{aligned}$$

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## Effect on atoms

Model of §7.5, electrons are harmonic oscillators,  
frequency  $\omega_j$ , damping  $\gamma_j$ , oscillator "strength"  $f_j$ , with  
 $\sum f_j = Z$ . Response to  $\vec{E}(\omega)$ :

$$\vec{x}_j(\omega) = -\frac{e}{m} \frac{\vec{E}(\omega)}{\omega_j^2 - \omega^2 - i\omega\gamma_j}.$$

From Jackson 7.51, the dielectric constant is

$$\epsilon(\omega) = 1 + \frac{4\pi Ne^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}.$$

Each of these electrons will absorb an energy

$$\begin{aligned} \Delta E &= -e \int_{-\infty}^{\infty} dt \vec{v}_j(t) \cdot \vec{E}(\vec{x}, t) \\ &= -\frac{e}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega (-i\omega x_j(\omega) e^{-i\omega t}) \int_{-\infty}^{\infty} d\omega' \vec{E}^*(\omega') e^{i\omega' t} \end{aligned}$$

The  $\int dt$  gives  $2\pi\delta(\omega - \omega')$  so  $\int d\omega'$  is trivial.

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$$\Delta E = ie \int_{-\infty}^{\infty} d\omega \omega x_j(\omega) \vec{E}^*(\omega) = 2e \text{Re} \int_0^{\infty} d\omega i\omega x_j(\omega) \vec{E}^*(\omega).$$

where because  $\vec{x}(t)$  and  $\vec{E}(t)$  are real,  $\vec{x}(-\omega) = \vec{x}^*(\omega)$ ,  
 $\vec{E}(-\omega) = \vec{E}^*(\omega)$ , and  $\int_{-\infty}^0$  can be folded into  $\int_0^{\infty}$ .

Take  $\vec{v} \parallel x$ , look at atom at  $(0, b, 0)$ , which feels

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}} \int d^3k \vec{E}(\vec{k}, \omega) e^{ik_2 b}.$$

The energy absorbed by this atom is

$$-\Delta E = \frac{2e^2}{m} \sum_j f_j \text{Re} \int_0^{\infty} d\omega \frac{i\omega |\vec{E}|^2}{\omega_j^2 - \omega^2 - i\omega\gamma_j},$$

and as there are  $2\pi N b db$  atoms per unit distance along  
the particle's path, the energy loss per unit distance is

$$\begin{aligned} \frac{dE}{dx} &= \int_0^{\infty} b db \text{Re} \int_0^{\infty} d\omega i\omega |\vec{E}|^2 \frac{4\pi N e^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \\ &= \int_0^{\infty} b db \text{Re} \int_0^{\infty} d\omega i\omega |\vec{E}|^2 (\epsilon(\omega) - 1). \end{aligned}$$

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$$\begin{aligned} \vec{E}(\vec{x} = (0, b, 0), \omega) &= \frac{-i}{(2\pi)^{3/2}} \int d^3k e^{ik_2 b} \left( \vec{k} - \frac{\omega\epsilon(\omega)}{c^2}\vec{v} \right) \frac{2ze}{\epsilon(\omega)} \frac{\delta(\omega - k_1\vec{v})}{k^2 - \omega^2\epsilon(\omega)/c^2} \\ &= \frac{-i2ze}{(2\pi)^{3/2} v \epsilon(\omega)} \int_{-\infty}^{\infty} dk_2 e^{ik_2 b} \int_{-\infty}^{\infty} dk_3 \\ &\quad \left( \vec{k} - \frac{\omega\epsilon(\omega)}{c^2}\vec{v} \right) \frac{1}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2\epsilon(\omega)/c^2}, \end{aligned}$$

where  $k_1 = \omega/v$ . For  $E_1$  this gives

$$\begin{aligned} E_1(\omega) &= \frac{-i2ze\omega}{(2\pi)^{3/2} v^2 \epsilon(\omega)} (1 - \epsilon(\omega)\beta^2) \int_{-\infty}^{\infty} dk_2 e^{ik_2 b} \\ &\quad \int_{-\infty}^{\infty} dk_3 \frac{1}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2\epsilon(\omega)/c^2} \\ &= \frac{-i2ze\omega}{\sqrt{2\pi} v^2 \epsilon(\omega)} (1 - \epsilon(\omega)\beta^2) \int_{-\infty}^{\infty} dk_2 e^{ik_2 b} \frac{1}{\sqrt{k_2^2 + \lambda^2}} \end{aligned}$$

$$\text{where } \lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2\epsilon(\omega)}{c^2} = \frac{\omega^2}{v^2} (1 - \beta^2\epsilon(\omega)).$$

Note whenever necessary  $\epsilon$  should be considered to have a positive imaginary part. This can be evaluated<sup>2</sup>

$$E_1(\omega) = -i\sqrt{\frac{2}{\pi}} \frac{ze\omega}{v^2} \left( \frac{1}{\epsilon(\omega)} - \beta^2 \right) K_0(\lambda b).$$

Next, we turn to  $E_2$  and  $E_3$ . First

$$\begin{aligned} E_2(\omega) &= \frac{-ize}{\sqrt{2\pi v\epsilon(\omega)}} \int_{-\infty}^{\infty} dk_2 e^{ik_2 b} k_2 \frac{1}{\sqrt{\lambda^2 + k_2^2}} \\ &= \frac{ze}{v} \sqrt{\frac{2}{\pi}} \frac{\lambda}{\epsilon(\omega)} K_1(\lambda b) \end{aligned}$$

<sup>2</sup>Abramowitz and Stegun tell us  $K_\nu(xz) = \frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\sqrt{\pi x^\nu}} \int_0^\infty \frac{\cos(xt) dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}$ . Expand the cosine in exponentials and rewrite the second term as the extension of the first for  $\infty < t < 0$ , to get  $\int_{-\infty}^\infty dx e^{ibx} (x^2 + \lambda^2)^{-1/2} = 2K_0(\lambda b)$ . The same integral with an extra  $x$  (or  $k_2$ ) in the integrand can be found as the derivative with respect to  $b$ , which is  $2i\lambda K_1(\lambda b)$ , as  $K'_0(z) = -K_1(z)$  (9.6.27).

$$E_3(\omega) = \frac{-ize}{\sqrt{2\pi v\epsilon(\omega)}} \int_{-\infty}^{\infty} dk_2 e^{ik_2 b} \int_{-\infty}^{\infty} \frac{k_3 dk_3}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2\epsilon(\omega)/c^2} = 0$$

where  $E_3 = 0$  by symmetry.

The energy loss due to impact parameters larger than  $b_0$  is

$$\begin{aligned} \left( \frac{dE}{dx} \right)_{b>b_0} &= \int_{b_0}^{\infty} b db \operatorname{Re} \int_0^\infty -i\omega\epsilon(\omega) |\vec{E}(\omega)|^2 d\omega \\ &= \frac{2}{\pi} \frac{z^2 e^2}{v^2} \operatorname{Re} \int_0^\infty d\omega (-i\omega)\epsilon(\omega) \int_{b_0}^{\infty} b db \\ &\quad \left[ \frac{\omega^2}{v^2} \left( \frac{1}{\epsilon(\omega)} - \beta^2 \right)^2 K_0^2(\lambda b) + \frac{\lambda^2}{\epsilon^2(\omega)} K_1^2(\lambda b) \right] \end{aligned}$$

The term in [ ] is

$$\left( \frac{1}{\epsilon(\omega)} - \beta^2 \right) \frac{\omega^2}{v^2 \epsilon(\omega)} [(1 - \beta^2 \epsilon(\omega)) K_0^2 - K_1^2]$$

The integral over impact parameter  $b$  can be done, as

$$\begin{aligned} \int_a^\infty x dx K_0^2(x) &= \frac{1}{2} a^2 (K_1^2(a) - K_0^2(a)) \\ \int_a^\infty x dx K_1^2(x) &= \frac{1}{2} a^2 (K_0^2(a) - K_1^2(a)) + a K_0(a) K_1(a). \end{aligned}$$

I don't quite get this, but Jackson claims

$$\left( \frac{dE}{dx} \right)_{b>b_0} = \frac{2}{\pi} \frac{z^2 e^2}{v^2} \operatorname{Re} \int_0^\infty d\omega (i\omega\lambda^* a) K_1(\lambda^* a) K_0(\lambda a) \left( \frac{1}{\epsilon(\omega)} - \beta^2 \right).$$

This evaluation is better than the free-electron one for large impact parameter  $b \gg a$ , but not for  $b \leq a$ . Below some cutoff  $b_0$  we use the previous free-electron calculation with  $\epsilon$  the energy loss corresponding to  $b_0$ .

From (1),

$$\begin{aligned} d\sigma &= 2\pi b db = 2\pi \sin\theta d\theta \left( \frac{ze^2}{2vp} \right)^2 \frac{1}{\sin^4(\theta/2)} \\ &= 2\pi \frac{dQ^2}{2p^2} \left( \frac{ze^2}{2vp} \right)^2 \left( \frac{4p^2}{Q^2} \right)^2 = 2\pi \frac{4p^2}{m} \frac{dT}{T^2} \left( \frac{ze^2}{2vp} \right)^2 \end{aligned}$$

$$\text{so } b^2 = \left( \frac{2ze^2}{v} \right)^2 \frac{1}{2mT}.$$

[Note: the above is my own, Jackson doesn't discuss this.]