Lecture 19 April 8, 2010

Charged particle Energy Loss

Fast heavy ($\gg m_e$) charged particle interacting with matter.
Collisions with electrons transfers lots of energy, not much deviation of particle.
Collisions with nuclei, if $m \ll m_{\rm Nucl}$, scattering important but not much energy loss.

Consider first the scattering with electrons, of a projectile with v, M, and $q = ze$, with $E = M\gamma c^2$, $P = M\beta\gamma c$.
Ignore binding of electron in atom, and its initial velocity.
It has mass m and charge $-e$. $M \gg m$, so in projectile's rest frame, electron Coulomb scatters, with the "well-known Rutherford scattering" cross section[1].

[1] See apology in lecture notes

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2vp}\right)^2 \frac{1}{\sin^4(\theta/2)}, \tag{1}$$

where $p = m\beta\gamma c$ is the momentum of the electron.
We want the change of the projectile's momentum, so need cross section in terms of momentum transfer rather than angle:
$Q^2 = -(p'^{\mu} - p^{\mu})^2 > 0$, which for elastic scattering will be

$$Q^2 = 4p^2 \sin^2(\theta/2), \quad dQ^2 = 2p^2 \sin\theta \, d\theta,$$

so

$$d\Omega = 2\pi \sin\theta \, d\theta = \frac{\pi}{p^2} dQ^2, \qquad \frac{d\sigma}{dQ^2} = \frac{\pi}{p^2}\frac{d\sigma}{d\Omega} = 4\pi \left(\frac{ze^2}{vQ^2}\right)^2.$$

Note I didn't Lorentz transform the cross section, because $d\sigma$ is an area transverse to the relative velocity back to the lab frame.

In projectile's frame, $P^\mu = (Mc, \vec{0})$, $p^\mu = (mc\gamma, -m\gamma\vec{v})$, so $P \cdot p = Mmc^2\gamma$, so $\beta^2 = (Mmc^2/P \cdot p)^2$. Energy lost to electron is $T = (p'^0 - p^0)c$ in the lab, where $p^\mu = (mc, \vec{0})$, so $mT = p \cdot (p' - p) = p \cdot p' - p^2 = -\frac{1}{2}(p' - p)^2 = \frac{1}{2}Q^2$. Replacing Q^2 by $2mT$ on both sides of the cross section equation,

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 e^4}{mv^2T^2}.$$

This formula will tell us how rapidly a swift projectile looses energy, but its validity is limited:

1) $T = \dfrac{Q^2}{2m} = 2\dfrac{p^2}{m}\sin^2\left(\dfrac{\theta}{2}\right) \le 2m(c\beta\gamma)^2$, so the cross section for $T > T_{\max} := 2m(c\beta\gamma)^2$ is zero. 2) Lower bound: Unless the projectile gives up enough energy to free the electron from the atom (or at least raise it to a higher quantum state), no energy will be lost, and the cross section should be zero. Call this energy ϵ.

Energy loss dE/dx

Material has N atoms/unit volume, Z electrons/atom. Projectile looses energy $T \pm dT/2$ for each of $ZN dx d\sigma/dT$ electrons it scatters off with that dT, so

$$
\begin{aligned}
-\frac{dE}{dx} &= NZ \int_\epsilon^{T\text{max}} T \frac{d\sigma}{dT}\, dT = 2\pi NZ \frac{z^2 e^4}{mv^2} \int_\epsilon^{T\text{max}} \frac{1}{T}\, dT \\
&= 2\pi NZ \frac{z^2 e^4}{mv^2} \ln\left(\frac{T\text{max}}{\epsilon}\right) \\
&= 2\pi NZ \frac{z^2 e^4}{mv^2} \ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right).
\end{aligned}
$$

Lots of corrections to this:

Dirac spin $\ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right) \to \ln\left(\frac{2mv^2\gamma^2}{\epsilon}\right) - \beta^2$.

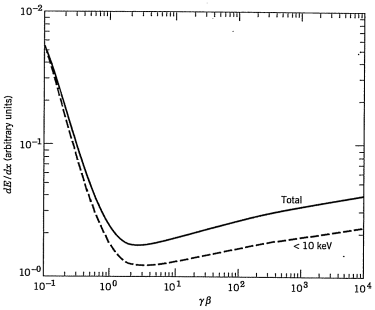
Energy loss $< \epsilon$ is not negligible, in fact it doubles dE/dx.

Features of dE/dx

But basic features given correctly:

- For $\beta \ll 1$, proportional to $1/v^2$, with coefficient $\propto NZ$ or the material's density. So loss per gram/cm^2 is roughly material-independent.

- For $\beta \sim 1$, grows logarithmically. Therefore a minimum ionizing value, somewhere around $\beta\gamma = 3$.



dE/dx from Jackson

Coherent Effects

[Note: we are skipping section 2.]

For $b \gg a \approx N^{-1/3}$ interaction with polarizable medium more appropriate than incoherent scattering by individual atoms.

Assume $\epsilon(\omega)$ but $\mu = 1$ for material. Maxwell's laws (in Gaussian units) become:

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= 4\pi\rho && (2) \\ \vec{\nabla} \cdot \vec{B} &= 0 && (3) \\ \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} &= \frac{4\pi}{c}\vec{J} && (4) \\ \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= 0 && (5) \end{aligned}$$

with

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

Fourier transforming

Fourier transform everything:

$$F(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3k \int d\omega F(\vec{k},\omega) e^{i\vec{k}\cdot\vec{x}-i\omega t},$$

we get

$$\begin{aligned}
\vec{E}(\vec{k},\omega) &= -i\vec{k}\Phi(\vec{k},\omega) + \frac{i\omega}{c}\vec{A}(\vec{k},\omega) \\
\vec{D}(\vec{k},\omega) &= -i\epsilon(\omega)\vec{k}\,\Phi(\vec{k},\omega) + \frac{i\omega\epsilon(\omega)}{c}\vec{A}(\vec{k},\omega) \\
\vec{B}(\vec{k},\omega) &= i\vec{k}\times\vec{A}(\vec{k},\omega)
\end{aligned}$$

so (2) and (4) become

$$\begin{aligned}
\epsilon(\omega)k^2\Phi(\vec{k},\omega) - \frac{\omega\epsilon(\omega)}{c}\vec{k}\cdot\vec{A}(\vec{k},\omega) &= 4\pi\rho(\vec{k},\omega) \\
-\vec{k}\times\left(\vec{k}\times\vec{A}(\vec{k},\omega)\right) + \frac{\omega}{c}\epsilon(\omega)\vec{k}\,\Phi(\vec{k},\omega) & \\
-\frac{\omega^2\epsilon(\omega)}{c^2}\vec{A}(\vec{k},\omega) &= \frac{4\pi}{c}\vec{J}(\vec{k},\omega)
\end{aligned}$$

Gauge invariance: $A^\mu \to A^\mu - \partial^\mu \Lambda$, or $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$, $\Phi \to \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$, is OK even in materials, so choose *modified* Lorenz condition

$$\frac{\epsilon}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0, \quad \text{or } \vec{k}\cdot\vec{A}(\vec{k},\omega) = \frac{\omega\epsilon(\omega)}{c}\Phi.$$

Then we can write our equations as

$$\begin{aligned} \epsilon(\omega)k^2\Phi(\vec{k},\omega) - \frac{\omega^2\epsilon^2(\omega)}{c^2}\Phi(\vec{k},\omega) &= 4\pi\rho(\vec{k},\omega) \\ k^2\vec{A}(\vec{k},\omega) - \frac{\omega^2\epsilon(\omega)}{c^2}\vec{A}(\vec{k},\omega) &= \frac{4\pi}{c}\vec{J}(\vec{k},\omega) \end{aligned}$$

Interaction at large b

Projectile with$\vec{v}$, will be essentially unchanged (or only slowly diminished)

$$\rho(\vec{x},t) = ze\delta^3(\vec{x}-\vec{v}t), \qquad \vec{J}((\vec{x},t) = \vec{v}\rho(\vec{x},t) = ze\vec{v}\delta^3(\vec{x}-\vec{v}t),$$

which means the fourier transformed source is

$$\begin{aligned} \rho(\vec{k},\omega) &= \frac{ze}{(2\pi)^2}\int d^3x dt\, \delta^3(\vec{x}-\vec{v}t)e^{-i\vec{k}\cdot\vec{x}+i\omega t} \\ &= \frac{ze}{(2\pi)^2}\int dt\, e^{-i(\vec{k}\cdot\vec{v}-\omega)t} = \frac{ze}{2\pi}\delta(\omega-\vec{k}\cdot\vec{v}) \end{aligned}$$

and $\vec{J}(\vec{k},\omega) = \vec{v}\rho(\vec{k},\omega)$. In Fourier space the equations for Φ and $\vec{A}$ become trivial to solve:

$$\Phi(\vec{k},\omega) = \frac{2ze}{\epsilon(\omega)}\frac{\delta(\omega-\vec{k}\cdot\vec{v})}{k^2-\omega^2\epsilon(\omega)/c^2}, \qquad \vec{A}(\vec{k},\omega) = \frac{\vec{v}\epsilon(\omega)}{c}\Phi(\vec{k},\omega)$$

$$\begin{aligned} \vec{E}(\vec{k},\omega) &= -i\vec{k}\Phi(\vec{k},\omega) + i\frac{\omega}{c}\vec{A}(\vec{k},\omega) \\ &= \left(-i\vec{k} + i\frac{\omega\epsilon(\omega)}{c^2}\vec{v}\right)\Phi(\vec{k},\omega). \end{aligned}$$

Effect on atoms

Model of §7.5, electrons are harmonic oscillators, frequency ω_j, damping γ_j, oscillator "strength" f_j, with $\sum f_j = Z$. Response to $\vec{E}(\omega)$:

$$\vec{x}_j(\omega) = -\frac{e}{m}\frac{\vec{E}(\omega)}{\omega_j^2 - \omega^2 - i\omega\gamma_j}.$$

From Jackson 7.51, the dielectric constant is

$$\epsilon(\omega) = 1 + \frac{4\pi N e^2}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}.$$

Each of these electrons will absorb an energy

$$\begin{aligned}\Delta E &= -e\int_{-\infty}^{\infty} dt\, \vec{v}_j(t)\cdot\vec{E}(\vec{x},t)\\ &= -\frac{e}{2\pi}\int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} d\omega\,\left(-i\omega x_j(\omega)e^{-i\omega t}\right)\int_{-\infty}^{\infty} d\omega'\,\vec{E}^*(\omega')e^{i\omega' t}\end{aligned}$$

The $\int dt$ gives $2\pi\delta(\omega-\omega')$ so $\int d\omega'$ is trivial.

$$\Delta E = ie\int_{-\infty}^{\infty} d\omega\, \omega x_j(\omega)\vec{E}^*(\omega) = 2e\,\mathrm{Re}\int_0^{\infty} d\omega\, i\omega x_j(\omega)\vec{E}^*(\omega).$$

where because $\vec{x}(t)$ and $\vec{E}(t)$ are real, $\vec{x}(-\omega) = \vec{x}^*(\omega)$, $\vec{E}(-\omega) = \vec{E}^*(\omega)$, and $\int_{-\infty}^0$ can be folded into $\int_0^\infty$.
Take $\vec{v} \parallel x$, look at atom at $(0, b, 0)$, which feels
$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\vec{E}(\vec{k},\omega)e^{ik_2 b}$.
The energy absorbed by this atom is

$$-\Delta E = \frac{2e^2}{m}\sum_j f_j\,\mathrm{Re}\int_0^\infty d\omega\,\frac{i\omega|\vec{E}|^2}{\omega_j^2 - \omega^2 - i\omega\gamma_j},$$

and as there are $2\pi Nbdb$ atoms per unit distance along the particle's path, the energy loss per unit distance is

$$\begin{aligned}\frac{dE}{dx} &= \int_0^\infty b\,db\,\mathrm{Re}\int_0^\infty d\omega\, i\omega|\vec{E}|^2\frac{4\pi Ne^2}{m}\sum_j\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}\\ &= \int_0^\infty b\,db\,\mathrm{Re}\int_0^\infty d\omega\, i\omega|\vec{E}|^2\left(\epsilon(\omega) - 1\right).\end{aligned}$$

$$
\begin{aligned}
&\vec{E}(\vec{x} = (0, b, 0), \omega) \\
&\quad = \frac{-i}{(2\pi)^{3/2}} \int d^3k e^{ik_2 b} \left(\vec{k} - \frac{\omega \epsilon(\omega)}{c^2} \vec{v} \right) \frac{2ze}{\epsilon(\omega)} \frac{\delta(\omega - k_1 \vec{v})}{k^2 - \omega^2 \epsilon(\omega)/c^2} \\
&\quad = \frac{-i2ze}{(2\pi)^{3/2} v \epsilon(\omega)} \int_{-\infty}^{\infty} dk_2\, e^{ik_2 b} \int_{-\infty}^{\infty} dk_3 \\
&\qquad \left(\vec{k} - \frac{\omega \epsilon(\omega)}{c^2} \vec{v} \right) \frac{1}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2 \epsilon(\omega)/c^2},
\end{aligned}
$$

where $k_1 = \omega/v$. For E_1 this gives

$$
\begin{aligned}
E_1(\omega) &= \frac{-i2ze\omega}{(2\pi)^{3/2} v^2 \epsilon(\omega)} \left(1 - \epsilon(\omega)\beta^2\right) \int_{-\infty}^{\infty} dk_2\, e^{ik_2 b} \\
&\qquad \int_{-\infty}^{\infty} dk_3 \frac{1}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2 \epsilon(\omega)/c^2} \\
&= \frac{-ize\omega}{\sqrt{2\pi} v^2 \epsilon(\omega)} \left(1 - \epsilon(\omega)\beta^2\right) \int_{-\infty}^{\infty} dk_2\, e^{ik_2 b} \frac{1}{\sqrt{k_2^2 + \lambda^2}}
\end{aligned}
$$

$$
\text{where} \quad \lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2 \epsilon(\omega)}{c^2} = \frac{\omega^2}{v^2} \left(1 - \beta^2 \epsilon(\omega)\right).
$$

Note whenever necessary ϵ should be considered to have a positive imaginary part. This can be evaluated[2]

$$E_1(\omega) = -i\sqrt{\frac{2}{\pi}}\frac{ze\omega}{v^2}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)K_0(\lambda b).$$

Next, we turn to E_2 and E_3. First

$$\begin{aligned} E_2(\omega) &= \frac{-ize}{\sqrt{2\pi}v\epsilon(\omega)}\int_{-\infty}^{\infty} dk_2\, e^{ik_2 b}\, k_2 \frac{1}{\sqrt{\lambda^2+k_2^2}} \\ &= \frac{ze}{v}\sqrt{\frac{2}{\pi}}\frac{\lambda}{\epsilon(\omega)}K_1(\lambda b) \end{aligned}$$

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[2] Abramowitz and Stegun tell us $K_\nu(xz) = \frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\sqrt{\pi}x^\nu}\int_0^\infty \frac{\cos(xt)dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}$. Expand the cosine in exponentials and rewrite the second term as the extension of the first for $\infty < t < 0$, to get $\int_{-\infty}^{\infty} dx e^{ibx}(x^2+\lambda^2)^{-1/2} = 2K_0(\lambda b)$. The same integral with an extra x (or k_2) in the integrand can be found as the derivative with respect to b, which is $2i\lambda K_1(\lambda b)$, as $K_0'(z) = -K_1(z)$ (9.6.27).

$$E_3(\omega) = \frac{-ize}{\sqrt{2\pi}v\epsilon(\omega)}\int_{-\infty}^{\infty} dk_2\, e^{ik_2 b}$$
$$\int_{-\infty}^{\infty}\frac{k_3\, dk_3}{\omega^2/v^2 + k_2^2 + k_3^2 - \omega^2\epsilon(\omega)/c^2} = 0$$

where $E_3 = 0$ by symmetry.

The energy loss due to impact parameters larger than b_0 is

$$\left(\frac{dE}{dx}\right)_{b>b_0} = \int_{b_0}^{\infty} bdb\mathrm{Re}\int_0^{\infty} -i\omega\epsilon(\omega)|\vec{E}(\omega)|^2 d\omega$$
$$= \frac{2}{\pi}\frac{z^2e^2}{v^2}\mathrm{Re}\int_0^{\infty} d\omega\,(-i\omega)\epsilon(\omega)\int_{b_0}^{\infty} b\, db$$
$$\left[\frac{\omega^2}{v^2}\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)^2 K_0^2(\lambda b) + \frac{\lambda^2}{\epsilon^2(\omega)}K_1^2(\lambda b)\right]$$

The term in [ ] is

$$\left(\frac{1}{\epsilon(\omega)} - \beta^2\right)\frac{\omega^2}{v^2\epsilon(\omega)}\left[\left(1 - \beta^2\epsilon(\omega)\right)K_0^2 - K_1^2\right]$$

The integral over impact parameter b can be done, as

$$\begin{aligned}\int_a^\infty x\,dx K_0^2(x) &= \frac{1}{2}a^2\left(K_1^2(a) - K_0^2(a)\right)\\ \int_a^\infty x\,dx K_1^2(x) &= \frac{1}{2}a^2\left(K_0^2(a) - K_1^2(a)\right) + aK_0(a)K_1(a).\end{aligned}$$

I don't quite get this, but Jackson claims

$$\left(\frac{dE}{dx}\right)_{b>b_0} = \frac{2}{\pi}\frac{z^2e^2}{v^2}\mathrm{Re}\int_0^\infty d\omega\,(i\omega\lambda^* a)K_1(\lambda^* a)K_0(\lambda a)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right).$$

This evaluation is better than the free-electron one for large impact parameter $b \gg a$, but not for $b \le a$. Below some cutoff b_0 we use the previous free-electron calculation with ϵ the energy loss corresponding to b_0.

From (1),

$$\begin{aligned} d\sigma &= 2\pi b db = 2\pi \sin\theta d\theta \left(\frac{ze^2}{2vp}\right)^2 \frac{1}{\sin^4(\theta/2)} \\ &= 2\pi \frac{dQ^2}{2p^2} \left(\frac{ze^2}{2vp}\right)^2 \left(\frac{4p^2}{Q^2}\right)^2 = 2\pi \frac{4p^2}{m} \frac{dT}{T^2} \left(\frac{ze^2}{2vp}\right)^2 \end{aligned}$$

so $b^2 = \left(\frac{2ze^2}{v}\right)^2 \frac{1}{2mT}$.

[Note: the above is my own, Jackson doesn't discuss this.]