Lecture 18 April 5, 2010

Darwin Lagrangian

Particle dynamics: $\vec{x}_j(t)$ evolves by $\vec{F}_{k\to j}(\vec{x}_j(t),\vec{x}_k(t))$, depends on where other particles are *at the same instant*. Violates relativity!
If the forces are given by a potential energy $V(\vec{x}_j(t),\vec{x}_k(t))$, that also violates relativity, unless $V\propto\delta(\vec{x}_j-\vec{x}_k)$. Not very useful.
But we know how to treat charged particles interacting electromagnetically if they are not moving too fast. We learned as freshmen how to do the lowest order ($c\to\infty$):

$$V(\vec{x}_j,\vec{x}_k)=\frac{q_jq_k}{|\vec{x}_j-\vec{x}_k|}\quad\text{and}\quad T=\frac{1}{2}\sum m_j\vec{v}_j^{\,2}.$$

This encapsulates the effect of the $\vec{E}$ one particle produces on the other.

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To next order

In our relativistic treatment

$$L_{\text{int}}=\sum_j q_j\left(-\Phi(\vec{x}_j)+\frac{1}{c}\vec{u}_j\cdot\vec{A}(\vec{x}_j)\right),$$

and $\Phi(\vec{x}_j)=\sum_k\dfrac{q_k}{|\vec{x}_j-\vec{x}_k|}$ is the $c\to\infty$ limit for the scalar potential.

Magnetic forces require moving particles to be produced, and moving particles to feel their effect. So these are v^2/c^2 effects. To this order, Φ and $\vec{A}$ depend on choice of gauge. Choose Coulomb ($\vec{\nabla}\cdot\vec{A}=0$), not Lorenz, because then $\nabla^2\Phi=-4\pi\rho$, and Φ is determined by instantaneous information: $\Phi(\vec{r},t)=\displaystyle\int d^3r'\frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}$ to all orders in v/c!

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Magnetic interaction

From $\partial_\sigma F^{\sigma j}=4\pi J^j/c$ we have

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}-\nabla^2\vec{A}+\vec{\nabla}\left(\frac{1}{c}\frac{\partial}{\partial t}\Phi+\vec{\nabla}\cdot\vec{A}\right)=4\pi\vec{J}/c.$$

The $\vec{\nabla}\cdot\vec{A}$ is zero in Coulomb gauge. Working accurate to order $(v/c)^2$ we may drop the $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$ term, as $\vec{A}$ is already order $(v/c)^1$. Thus we may take

$$\nabla^2\vec{A}=-\frac{4\pi}{c}\vec{J}+\frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\Phi.$$

Particle j contributes
$q_j\vec{v}_j\delta^3(\vec{x}\,'-\vec{x}_j)$ to $\vec{J}(\vec{x}\,')$ and $\dfrac{q_j}{|\vec{x}\,'-\vec{x}\,'_j|}$ to $\Phi(\vec{x}\,')$,
so it contributes $q_j\dfrac{\vec{v}_j\cdot(\vec{x}\,'-\vec{x}_j)}{|\vec{x}\,'-\vec{x}_j|^3}$ to $\dfrac{\partial\Phi}{\partial t}$.
The Green's function for Laplace's equation is $1/|\vec{x}-\vec{x}\,'|$, which we apply to the right hand side:

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$$\begin{aligned}\vec{A}(\vec{x})&=\int\frac{d^3x'}{|\vec{x}-\vec{x}\,'|}\left(\frac{1}{c}\vec{J}(\vec{x}\,')-\frac{1}{4\pi c}\vec{\nabla}'\frac{\partial}{\partial t}\Phi(\vec{x}\,')\right)\\&=\int\frac{d^3x'}{|\vec{x}-\vec{x}\,'|}\left[\frac{q_jv_j}{c}\delta^3(\vec{x}\,'-\vec{x}_j)\right.\\&\qquad\left.-\frac{q_j}{4\pi c}\vec{\nabla}'\left(\frac{\vec{v}_j\cdot(\vec{x}\,'-\vec{x}_j)}{|\vec{x}\,'-\vec{x}_j|^3}\right)\right]\\&=\frac{q_j\vec{v}_j}{c|\vec{x}-\vec{x}_j|}+\frac{q_j}{4\pi c}\int d^3x'\left(\frac{\vec{v}_j\cdot(\vec{x}\,'-\vec{x}_j)}{|\vec{x}\,'-\vec{x}_j|^3}\right)\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}\,'|}\end{aligned}$$

where we have integrated by parts and thrown away the surface at infinity. The gradient $\vec{\nabla}'\sim-\vec{\nabla}$ action on a function of $\vec{x}-\vec{x}\,'$, so we can pull $\vec{\nabla}$ out of the integral. Let $\vec{r}=\vec{x}-\vec{x}_j$ and $\vec{y}=\vec{x}\,'-\vec{x}_j$. Then

$$\vec{A}(\vec{x})=\frac{q_j\vec{v}_j}{c|\vec{r}|}-\frac{q_j}{4\pi c}\vec{\nabla}\int d^3y\frac{\vec{v}_j\cdot\vec{y}}{|\vec{y}|^3}\frac{1}{|\vec{y}-\vec{r}|}$$

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The integral can be done by choosing $z\parallel\vec{r}$ and $\vec{v}_j$ in the xz plane:

$$\begin{aligned}&\int d^3y\frac{\vec{v}_j\cdot\vec{y}}{|\vec{y}|^3}\frac{1}{|\vec{y}-\vec{r}|}\\&=\int_0^\infty y^2dy\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi\\&\qquad\frac{y(\cos\theta v_{jz}+\sin\theta\cos\phi v_{jx})}{y^3}\frac{1}{\sqrt{y^2+r^2-2yr\cos\theta}}\end{aligned}$$

The ϕ integral kills the v_{jx} term and then what remains is $2\pi v_{jz}$ times

$$\int_0^\infty dy\int_{-1}^1du\frac{u}{\sqrt{y^2+r^2-2yru}}=1,$$

though this integral is not as straightforward as Jackson claims. Writing $v_{jz}=\vec{v}_j\cdot\vec{r}/r$, we have

$$\vec{A}(\vec{r})=\frac{q_j}{c}\left[\frac{\vec{v}_j}{|\vec{r}|}-\frac{1}{2}\vec{\nabla}\left(\frac{\vec{v}_j\cdot\vec{r}}{r}\right)\right].$$

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Applying the gradient, we get

$$\vec{A}_j(\vec{x}_k)=\frac{q_j}{2c|\vec{x}_j-\vec{x}_k|}\left[\vec{v}_j+\frac{(\vec{x}_k-\vec{x}_j)\vec{v}_j\cdot(\vec{x}_k-\vec{x}_j)}{|\vec{x}_k-\vec{x}_j|}\right].$$

Multiplying by $q_k\vec{v}_k/c$ to get the appropriate contribution to L_{int}, and correcting the free-particle Lagrangian, $-mc^2\gamma^{-1}+mc^2\approx\frac{1}{2}mv^2+\frac{1}{8}mv^4/c^2$, we get the Darwin Lagrangian

$$\begin{aligned}L_{\text{Darwin}}&=\frac{1}{2}\sum_j m_jv_j^2+\frac{1}{8c^2}\sum_j m_jv_j^4-\frac{1}{2}\sum_{j\neq k}\frac{q_jq_k}{r_{jk}}\\&\quad+\frac{1}{4c^2}\sum_{j\neq k}\frac{q_jq_k}{r_{jk}}\left[\vec{v}_j\cdot\vec{v}_k+(\vec{v}_j\cdot\hat{r}_{jk})(\vec{v}_k\cdot\hat{r}_{jk})\right],\end{aligned}$$

where of course $\vec{r}_{jk}:=\vec{x}_j-\vec{x}_k$, $r_{jk}:=|\vec{r}_{jk}|$, and $\hat{r}_{jk}=\vec{r}_{jk}/r_{jk}$.
This is used in atomic physics (with $\vec{v}\to\vec{\alpha}$ for Dirac) and in plasma physics.

The Proca Lagrangian

For Maxwell's electromagnetism:

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu A^\mu.$$

Does not give complete equations of motion A^μ.
Consider adding a term proportional to A^2:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{\mu^2}{8\pi} A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu,$$

known as the Proca Lagrangian. Still have $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$, not an independent field. $\therefore$ homogeneous Maxwell equations still hold (as $\mathbf{F} = \mathbf{d}A$). Extra term doesn't change ${P_\alpha}^\mu$ (no $\partial_\alpha A_\beta$ dependence), so change in equations of motion is just from $\partial\mathcal{L}/\partial A^\mu = (\mu^2/4\pi) A_\mu$, and

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha.$$

One consequence comes from taking the 4-divergence of this equation:

Proca Equations of Motion

$$\underbrace{\partial^\alpha \partial^\beta F_{\beta\alpha}}_{0} + \mu^2 \partial^\alpha A_\alpha = \frac{4\pi}{c} \underbrace{\partial^\alpha J_\alpha}_{0},$$

where the first vanishing is by symmetry and the second assumes charge is still conserved, to the continuity equation $\partial^\alpha J_\alpha = 0$ still holds. Thus $\partial^\alpha A_\alpha = 0$ is an equation of motion, not a gauge condition! Then

$$\partial^\beta F_{\beta\alpha} = \Box A_\alpha, \qquad \left(\Box + \mu^2\right) A_\alpha = \frac{4\pi}{c} J_\alpha.$$

In the absence of sources, this has solutions as before,

$$\sum_{\vec{k}} \left(A^\mu_{\vec{k}+} e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} t} + A^\mu_{\vec{k}-} e^{i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t} \right),$$

but with $\omega^2 = c^2(\vec{k}^2 + \mu^2)$.

Particle content

Quantum mechanically we know $\vec{p} = -i\hbar\vec{\nabla} \sim \hbar\vec{k}$ and $E = i\hbar\partial/\partial t = \pm\hbar\omega$, so $\omega^2 = c^2(\vec{k}^2 + \mu^2)$ tells us we have particles for which $E^2 = P^2c^2 + \mu^2\hbar^2c^2$. Of course quantum field theoriests take $\hbar = 1$ and $c = 1$, so this represents a massive photon with mass μ.

Static solution:

If we consider a point charge at rest and look for the static field it would generate, we need to solve

$$\nabla^2\Phi + \mu^2\Phi = -4\pi q \delta^3(\vec{r})$$

or

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\Phi}{\partial r} \right) + r^2\mu^2\Phi = -q\delta(r).$$

Away from $r = 0$ this clearly requires $r\Phi(r) = Ce^{-\mu r}$.

So $\Phi(r) = C\dfrac{e^{-\mu r}}{r}$, and Gauss's law tells us

$$-4\pi q = 4\pi R^2 \, d\Phi/dr|_R + \mu^2 \int_{r<R} \Phi \underset{R\to 0}{\longrightarrow} 4\pi C,$$

so $C = q$ and

$$\Phi(\vec{x}) = q\frac{e^{-\mu r}}{r}, \qquad \text{with} \quad r = |\vec{x}|.$$

This is the well-known Yukawa potential, which nuclear physicists had found was a good fit to the binding of nucleons in a nucleus, leading Yukawa to propose the existance of a massive carrier of the nuclear force, which we now know to be the π meson.

$\vec{A}$ in Superconductors

In the BCS theory of superconductivity, electrons form pairs, and each pair acts like a boson. So the quantum mechanical state that each pair is in can be multiply occupied, and superconductivity occurs when states develop macroscopic occupation numbers, $\gg 1$. The wave function $\psi(\vec{x})$ describing these particles is a complex function, with the density of particles $n(\vec{x}) = \psi^*\psi$, so $\psi = n(\vec{x})e^{i\theta(\vec{x})}$. We may approximate $n(\vec{x})$ as being roughly constant.
The velocity of these particles is related to the canonical momentum by

$$\vec{v} = \frac{1}{m}\left(\vec{P} - \frac{q}{c}\vec{A} \right)$$

which can be viewed as an operator acting between ψ^* and ψ. It is the *canonical momentum* $\vec{P}$ which acts like $-i\hbar\vec{\nabla}$. Thus the current density is

$$\vec{J} = q\psi^*\vec{v}\psi = \frac{nq}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A} \right).$$

If we take the curl of both sides of

$$\vec{J} = \frac{nq}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A} \right),$$

we get

$$\vec{\nabla} \times \vec{J} = -\frac{nq^2}{mc}\vec{\nabla} \times \vec{A} = -\frac{nq^2}{mc}\vec{B}, \tag{1}$$

as $\vec{\nabla} \times \vec{\nabla}\theta = 0$. This equation doesn't quite say

$$\vec{J} = -\frac{nq^2}{mc}\vec{A}, \tag{2}$$

but it does say, in a simply connected region, that the difference is the gradient of something, and as such a gradient could be added to $\vec{A}$ by a gauge transformation, we might as well assume (2), which is known as the London equation. This gauge is still compatible with Lorenz (which can be viewed as determining A^0), so we have

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} = \frac{4\pi nq^2}{mc^2}\vec{A},$$

which is the Proca equation with $\mu^2 = 4\pi nq^2/mc^2$.

London Penetration Depth

At the boundary of the superconductor, if no current is crossing the boundary, we must have $\vec{n} \cdot \vec{A} = 0$. If we look for a static solution for a planar boundary $\perp z$, uniform along the boundary, we have $A \propto e^{-\mu z}$. The London penetration depth is

$$\lambda_{\mathrm{L}} := \frac{1}{\mu} = \sqrt{\frac{mc^2}{4\pi n q^2}}.$$

With $q = -2e$ and $m = 2m_e$ for the electron pair, and taking n as the density of valence electrons, the penetration depth is of the order of tens of nanometers.

As the A field is not penetrating further than that into the medium, any external magnetic field has been excluded.

Vortex Lines

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But magnetic field lines can enter the medium if our assumption of being able to do away with $\vec{\nabla} \cdot \vec{A}$ by a gauge transformation is not correct. That could happen if the region of the superconductor is not simply connected — that is, a flux line could enter and destroy the superconducting region around which θ is incremented by a multiple of 2π.
This is called a vortex line, and corresponds to a quantized amount of flux, as

$$\oint \vec{A} \cdot d\ell = 2\pi N \hbar c/q = \int_S \vec{\nabla} \times \vec{A} = \Phi_B, \quad \text{with } N \in \mathbb{Z}.$$

With $q = -2e$, the quantum of flux is $hc/2e$.