Lecture 18 April 5, 2010

Darwin Lagrangian

Particle dynamics: $\vec{x}_j(t)$ evolves by $\vec{F}_{k\to j}(\vec{x}_j(t), \vec{x}_k(t))$, depends on where other particles are *at the same instant.* Violates relativity!
If the forces are given by a potential energy $V(\vec{x}_j(t), \vec{x}_k(t))$, that also violates relativity, unless $V \propto \delta(\vec{x}_j - \vec{x}_k)$. Not very useful.
But we know how to treat charged particles interacting electromagnetically if they are not moving too fast. We learned as freshmen how to do the lowest order ($c \to \infty$):

$$V(\vec{x}_j, \vec{x}_k) = \frac{q_j q_k}{|\vec{x}_j - \vec{x}_k|} \quad \text{and} \quad T = \frac{1}{2}\sum m_j \vec{v}_j^{\,2}.$$

This encapsulates the effect of the $\vec{E}$ one particle produces on the other.

To next order

In our relativistic treatment

$$L_{\text{int}} = \sum_j q_j \left(-\Phi(\vec{x}_j) + \frac{1}{c} \vec{u}_j \cdot \vec{A}(\vec{x}_j) \right),$$

and $\Phi(\vec{x}_j) = \sum_k \frac{q_k}{|\vec{x}_j - \vec{x}_k|}$ is the $c \to \infty$ limit for the scalar potential.

Magnetic forces require moving particles to be produced, and moving particles to feel their effect. So these are v^2/c^2 effects. To this order, Φ and $\vec{A}$ depend on choice of gauge. Choose Coulomb ($\vec{\nabla} \cdot \vec{A} = 0$), not Lorenz, because then $\nabla^2 \Phi = -4\pi\rho$, and Φ is determined by instantaneous information: $\Phi(\vec{r}, t) = \int d^3r' \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$ to all orders in v/c!

Magnetic interaction

From $\partial_\sigma F^{\sigma j} = 4\pi J^j/c$ we have

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} + \vec{\nabla}\left(\frac{1}{c}\frac{\partial}{\partial t}\Phi + \vec{\nabla}\cdot\vec{A}\right) = 4\pi\vec{J}/c.$$

The $\vec{\nabla}\cdot\vec{A}$ is zero in Coulomb gauge. Working accurate to order $(v/c)^2$ we may drop the $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$ term, as $\vec{A}$ is already order $(v/c)^1$. Thus we may take

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J} + \frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\Phi.$$

Particle j contributes

$q_j\vec{v}_j\delta^3(\vec{x}\,' - \vec{x}_j)$ to $\vec{J}(\vec{x}\,')$ and $\dfrac{q_j}{|\vec{x}\,' - \vec{x}\,'_j|}$ to $\Phi(\vec{x}\,')$,

so it contributes $q_j\dfrac{\vec{v}_j\cdot(\vec{x}\,' - \vec{x}_j)}{|\vec{x}\,' - \vec{x}_j|^3}$ to $\dfrac{\partial\Phi}{\partial t}$.

The Green's function for Laplace's equation is $1/|\vec{x} - \vec{x}\,'|$, which we apply to the right hand side:

$$
\begin{aligned}
\vec{A}(\vec{x}) &= \int \frac{d^3x'}{|\vec{x}-\vec{x}\,'|}\left(\frac{1}{c}\vec{J}(\vec{x}\,') - \frac{1}{4\pi c}\vec{\nabla}\,'\frac{\partial}{\partial t}\Phi(\vec{x}\,')\right) \\
&= \int \frac{d^3x'}{|\vec{x}-\vec{x}\,'|}\left[\frac{q_j v_j}{c}\delta^3(\vec{x}\,'-\vec{x}_j)\right. \\
&\qquad\qquad\left. -\frac{q_j}{4\pi c}\vec{\nabla}\,'\left(\frac{\vec{v}_j\cdot(\vec{x}\,'-\vec{x}_j)}{|\vec{x}\,'-\vec{x}_j|^3}\right)\right] \\
&= \frac{q_j\vec{v}_j}{c|\vec{x}-\vec{x}_j|} + \frac{q_j}{4\pi c}\int d^3x'\left(\frac{\vec{v}_j\cdot(\vec{x}\,'-\vec{x}_j)}{|\vec{x}\,'-\vec{x}_j|^3}\right)\vec{\nabla}\,'\frac{1}{|\vec{x}-\vec{x}\,'|}
\end{aligned}
$$

where we have integrated by parts and thrown away the surface at infinity. The gradient $\vec{\nabla}\,' \sim -\vec{\nabla}$ action on a function of $\vec{x}-\vec{x}\,'$, so we can pull $\vec{\nabla}$ out of the integral. Let $\vec{r} = \vec{x}-\vec{x}_j$ and $\vec{y} = \vec{x}\,'-\vec{x}_j$. Then

$$
\vec{A}(\vec{x}) = \frac{q_j\vec{v}_j}{c|\vec{r}|} - \frac{q_j}{4\pi c}\vec{\nabla}\int d^3y\,\frac{\vec{v}_j\cdot\vec{y}}{|\vec{y}|^3}\frac{1}{|\vec{y}-\vec{r}|}
$$

The integral can be done by choosing $z \parallel \vec{r}$ and $\vec{v}_j$ in the xz plane:

$$\int d^3y \frac{\vec{v}_j \cdot \vec{y}}{|\vec{y}|^3} \frac{1}{|\vec{y} - \vec{r}|}$$
$$= \int_0^\infty y^2 dy \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi$$
$$\frac{y(\cos\theta v_{jz} + \sin\theta\cos\phi v_{jx})}{y^3} \frac{1}{\sqrt{y^2 + r^2 - 2yr\cos\theta}}$$

The ϕ integral kills the v_{jx} term and then what remains is $2\pi v_{jz}$ times

$$\int_0^\infty dy \int_{-1}^1 du \frac{u}{\sqrt{y^2 + r^2 - 2yru}} = 1,$$

though this integral is not as straightforward as Jackson claims. Writing $v_{jz} = \vec{v}_j \cdot \vec{r}/r$, we have

$$\vec{A}(\vec{r}) = \frac{q_j}{c} \left[\frac{\vec{v}_j}{|\vec{r}|} - \frac{1}{2} \vec{\nabla} \left(\frac{\vec{v}_j \cdot \vec{r}}{r} \right) \right].$$

Applying the gradient, we get

$$\vec{A}_j(\vec{x}_k) = \frac{q_j}{2c|\vec{x}_j - \vec{x}_k|}\left[\vec{v}_j + \frac{(\vec{x}_k - \vec{x}_j)\vec{v}_j \cdot (\vec{x}_k - \vec{x}_j)}{|\vec{x}_k - \vec{x}_j|}\right].$$

Multiplying by $q_k\vec{v}_k/c$ to get the appropriate contribution to L_{int}, and correcting the free-particle Lagrangian, $-mc^2\gamma^{-1} + mc^2 \approx \frac{1}{2}mv^2 + \frac{1}{8}mv^4/c^2$, we get the Darwin Lagrangian

$$\begin{aligned} L_{\text{Darwin}} &= \frac{1}{2}\sum_j m_j v_j^2 + \frac{1}{8c^2}\sum_j m_j v_j^4 - \frac{1}{2}\sum_{j\neq k}\frac{q_j q_k}{r_{jk}} \\ &\quad + \frac{1}{4c^2}\sum_{j\neq k}\frac{q_j q_k}{r_{jk}}\left[\vec{v}_j\cdot\vec{v}_k + (\vec{v}_j\cdot\hat{r}_{jk})(\vec{v}_k\cdot\hat{r}_{jk})\right], \end{aligned}$$

where of course $\vec{r}_{jk} := \vec{x}_j - \vec{x}_k$, $r_{jk} := |\vec{r}_{jk}|$, and $\hat{r}_{jk} = \vec{r}_{jk}/r_{jk}$.
This is used in atomic physics (with $\vec{v} \to \vec{\alpha}$ for Dirac) and in plasma physics.

The Proca Lagrangian

For Maxwell's electromagnetism:

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu A^\mu.$$

Does not give complete equations of motion A^μ.
Consider adding a term proportional to A^2:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{\mu^2}{8\pi} A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu,$$

known as the Proca Lagrangian. Still have $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$, not an independent field. $\therefore$ homogeneous Maxwell equations still hold (as $\mathbf{F} = \mathbf{d}A$). Extra term doesn't change $P_\alpha{}^\mu$ (no $\partial_\alpha A_\beta$ dependence), so change in equations of motion is just from $\partial\mathcal{L}/\partial A^\mu = (\mu^2/4\pi) A_\mu$, and

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c} J_\alpha.$$

One consequence comes from taking the 4-divergence of this equation:

Proca Equations of Motion

$$\underbrace{\partial^\alpha \partial^\beta F_{\beta\alpha}}_{0} + \mu^2 \partial^\alpha A_\alpha = \frac{4\pi}{c} \underbrace{\partial^\alpha J_\alpha}_{0},$$

where the first vanishing is by symmetry and the second assumes charge is still conserved, to the continuity equation $\partial^\alpha J_\alpha = 0$ still holds. Thus
$\partial^\alpha A_\alpha = 0$ is an equation of motion, not a gauge condition!
Then

$$\partial^\beta F_{\beta\alpha} = \Box A_\alpha, \qquad \left(\Box + \mu^2\right) A_\alpha = \frac{4\pi}{c} J_\alpha.$$

In the absence of sources, this has solutions as before,

$$\sum_{\vec{k}} \left(A^\mu_{\vec{k}\,+} e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} t} + A^\mu_{\vec{k}\,-} e^{i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t} \right),$$

but with $\omega^2 = c^2(\vec{k}^{\,2} + \mu^2)$.

Particle content

Quantum mechanically we know $\vec{p} = -i\hbar\vec{\nabla} \sim \hbar\vec{k}$ and $E = i\hbar\partial/\partial t = \pm\hbar\omega$, so $\omega^2 = c^2(\vec{k}^{\,2} + \mu^2)$ tells us we have particles for which $E^2 = P^2c^2 + \mu^2\hbar^2c^2$. Of course quantum field theoriests take $\hbar = 1$ and $c = 1$, so this represents a massive photon with mass μ.

Static solution:

If we consider a point charge at rest and look for the static field it would generate, we need to solve

$$\nabla^2\Phi + \mu^2\Phi = -4\pi q\delta^3(\vec{r})$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + r^2\mu^2\Phi = -q\delta(r).$$

Away from $r = 0$ this clearly requires $r\Phi(r) = Ce^{-\mu r}$.

So $\Phi(r) = C\frac{e^{-\mu r}}{r}$, and Gauss's law tells us

$$-4\pi q = 4\pi R^2 \, d\Phi/dr|_R + \mu^2 \int_{r<R} \Phi \xrightarrow[R\to 0]{} 4\pi C,$$

so $C = q$ and

$$\Phi(\vec{x}) = q\frac{e^{-\mu r}}{r}, \qquad \text{with} \quad r = |\vec{x}|.$$

This is the well-known Yukawa potential, which nuclear physicists had found was a good fit to the binding of nucleons in a nucleus, leading Yukawa to propose the existance of a massive carrier of the nuclear force, which we now know to be the π meson.

$\vec{A}$ in Superconductors

In the BCS theory of superconductivity, electrons form pairs, and each pair acts like a boson. So the quantum mechanical state that each pair is in can be multiply occupied, and superconductivity occurs when states develop macroscopic occupation numbers, $\gg 1$. The wave function $\psi(\vec{x})$ describing these particles is a complex function, with the density of particles $n(\vec{x}) = \psi^*\psi$, so $\psi = n(\vec{x})e^{i\theta(\vec{x})}$. We may approximate $n(\vec{x})$ as being roughly constant.

The velocity of these particles is related to the canonical momentum by

$$\vec{v} = \frac{1}{m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)$$

which can be viewed as an operator acting between ψ^* and ψ. It is the *canonical momentum* $\vec{P}$ which acts like $-i\hbar\vec{\nabla}$. Thus the current density is

$$\vec{J} = q\psi^*\vec{v}\psi = \frac{nq}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A}\right).$$

If we take the curl of both sides of

$$\vec{J} = \frac{nq}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A}\right),$$

we get

$$\vec{\nabla}\times\vec{J} = -\frac{nq^2}{mc}\vec{\nabla}\times\vec{A} = -\frac{nq^2}{mc}\vec{B}, \tag{1}$$

as $\vec{\nabla}\times\vec{\nabla}\theta = 0$. This equation doesn't quite say

$$\vec{J} = -\frac{nq^2}{mc}\vec{A}, \tag{2}$$

but it does say, in a simply connected region, that the difference is the gradient of something, and as such a gradient could be added to $\vec{A}$ by a gauge transformation, we might as well assume (2), which is known as the London equation. This gauge is still compatible with Lorenz (which can be viewed as determining A^0), so we have

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} = \frac{4\pi nq^2}{mc^2}\vec{A},$$

which is the Proca equation with $\mu^2 = 4\pi nq^2/mc^2$.

London Penetration Depth

At the boundary of the superconductor, if no current is crossing the boundary, we must have $\vec{n} \cdot \vec{A} = 0$. If we look for a static solution for a planar boundary $\perp z$, uniform along the boundary, we have $A \propto e^{-\mu z}$. The London penetration depth is

$$\lambda_{\mathrm{L}} := \frac{1}{\mu} = \sqrt{\frac{mc^2}{4\pi n q^2}}.$$

With $q = -2e$ and $m = 2m_e$ for the electron pair, and taking n as the density of valence electrons, the penetration depth is of the order of tens of nanometers.

As the A field is not penetrating further than that into the medium, any external magnetic field has been excluded.

Vortex Lines

But magnetic field lines can enter the medium if our assumption of being able to do away with $\vec{\nabla} \cdot \vec{A}$ by a gauge transformation is not correct. That could happen if the region of the superconductor is not simply connected — that is, a flux line could enter and destroy the superconducting region around which θ is incremented by a multiple of 2π.
This is called a vortex line, and corresponds to a quantized amount of flux, as

$$\oint \vec{A} \cdot d\ell = 2\pi N \hbar c/q = \int_S \vec{\nabla} \times \vec{A} = \Phi_B, \quad \text{with } N \in \mathbb{Z}.$$

With $q = -2e$, the quantum of flux is $hc/2e$.