Lecture 17 April 1, 2010

Canonical Momentum Density

We have seen that in field theory the Lagrangian is an integral of the Lagrangian density

$$\mathcal{L}(\phi_i, \partial\phi_i/\partial x^\nu, x^\xi)$$

and the equations of motion come from the functional derivatives of L with respect to the local values of the fields, but because the Lagrangian density is local, these are given by $\dfrac{\partial\mathcal{L}}{\partial\phi_i(x^\nu)}$ and $\dfrac{\partial\mathcal{L}}{\partial(\partial_\rho\phi_i(x^\nu))}$, which are functions of x^μ. The Euler-Lagrange equations involved not a total momentum but a momentum *density*. For a scalar field ϕ_j this would be

$$P_j^\mu(x^\rho) = \left.\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_j}\right|_{x^\rho}.$$

For electromagnetism we have not a scalar but four fields A^ν, so we have *four* 4-vector fields

$$P_\alpha{}^\mu := \frac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial A^\alpha(\vec{x},t)}{\partial x^\mu} \right)}.$$

Last time we saw that the lagrangian *density* for the electromagnetic fields is

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J_\mu A^\mu,$$

so the canonical momentum densities are

$$P_\alpha{}^\mu := \frac{\partial \mathcal{L}}{\partial(\partial A^\alpha(\vec{x},t)/\partial x^\mu)} = -\frac{1}{4\pi} F^\mu{}_\alpha,$$

because, as we saw last time, only the F^2 term depends on $\partial A^\alpha/\partial x^\mu$.

The Stress (Energy-Momentum) Tensor

Discrete mechanics: $H = \sum_i P_i \dot{q}_i - L$, $\dot{q}_i \to P_i$
Field theory: Hamiltonian density

$$\mathcal{H}(\vec{x}) := \sum_i P_i(\vec{x})\dot{\phi}_i(\vec{x}) - \mathcal{L}(\vec{x}) = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial\phi_i/\partial x^0)}\frac{\partial \phi_i}{\partial x^0} - \mathcal{L}.$$

Time has been picked out. More generally, let

$$T^\mu{}_\nu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial\phi_i/\partial x^\mu)}\frac{\partial \phi_i}{\partial x^\nu} - \delta^\mu_\nu \mathcal{L}.$$

This object goes by the names **energy-momentum tensor** or **stress-energy tensor** or **canonical stress tensor**, and we see the hamiltonian density is the 00 component of this tensor.

$T^{\mu}{}_{\nu}$ for electromagnetism

For electromagnetism, ϕ_i is replaced by A^λ,

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial A^\lambda/\partial x^\mu)} \frac{\partial A^\lambda}{\partial x^\nu} - \delta^\mu_\nu \mathcal{L}.$$

The first factor in the first term is

$$\frac{\partial \mathcal{L}}{\partial(\partial A^\lambda/\partial x^\mu)} = -\frac{1}{4\pi} F^{\mu}{}_{\lambda},$$

so our first (tentative) expression for the energy momentum tensor is

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(F_{\mu\lambda}\frac{\partial A^\lambda}{\partial x^\nu} - \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right).$$

This expression has good and bad properties!

We have seen T^{00} is the Hamiltonian density.

$\int T^{0i} = P^i$

If T^{00} is the energy density, is T^{0j} the density of momentum?

$$T^{0i} = \frac{1}{4\pi} F_{0\lambda} \partial_i A^\lambda = \frac{1}{4\pi} E_j \partial_i A_j = \frac{1}{4\pi} \left(\vec{E} \times \vec{B} + \left(\vec{E} \cdot \vec{\nabla} \right) \vec{A} \right)_i .$$

Poynting tells us the first term is the correct expression. The second term is unwanted, and also not gauge invariant.

But we haven't included charges in the momentum, and if no charges, $\vec{\nabla} \cdot \vec{E} = 0$, $\left(\vec{E} \cdot \vec{\nabla} \right) \vec{A}_i = \vec{\nabla} \cdot (A_i \vec{E}) - A_i \vec{\nabla} \cdot \vec{E}$ is a total derivative, and won't affect the *total* momentum.

So we do have the good property $\displaystyle\int d^3x T^{0\mu} = P^\mu$, the total momentum. But we don't have the right density.

Conservation of Momentum

If $\mathcal{L}$ has no *explicit* dependence on x^μ, we can show $\partial_\mu T^\mu{}_\nu = 0$, where ∂_μ is the stream derivative. For

$$\partial_\mu T^\mu{}_\nu = \sum_i \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \partial_\nu \phi_i + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \partial_\nu \phi_i - \partial_\nu \mathcal{L}.$$

The derivative in the last term is given by the chain rule

$$-\partial_\nu \mathcal{L} = -\sum_i \frac{\partial \mathcal{L}}{\partial \phi_i} \partial_\nu \phi_i - \sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} \partial_\nu \partial_\mu \phi_i$$

so

$$\partial_\mu T^\mu{}_\nu = \sum_i \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} \right) \partial_\nu \phi_i$$

and the parenthesis vanishes by the equations of motion. Thus we have

$$\partial_\mu T^\mu{}_\nu = 0. \tag{1}$$

This is a good thing.

Non-symmetry is a bad thing

Note: this $T^{\mu\nu} \neq T^{\nu\mu}$. This is a problem. We expect the angular momentum density to be given by $\epsilon_{ijk} x_j T^{0k}$, but that requires $T^{\mu\nu}$ to be symmetric.

So our $T^{\mu\nu}$ is good at giving the total momentum and being conserved, but bad in not being symmetric or gauge-invariant. Need modification keeping good properties but changing bad ones.

If we have a tensor $\psi^{\rho\mu\nu} = -\psi^{\mu\rho\nu}$ (antisymmetric in first two indices) and we add $\partial_\rho \psi^{\rho\mu\nu}$ to $T^{\mu\nu}$,

$$\Delta\left(\partial_\mu T^{\mu\nu}\right) = \partial_\mu \partial_\rho \psi^{\rho\mu\nu} = 0,$$

so the new $T^{\mu\nu}$ is also conserved. Furthermore,

$$\int d^3x \Delta T^{0\nu} = \int d^3x \partial_\rho \psi^{\rho 0\nu} = \int d^3x \partial_j \psi^{j0\nu} = \int_S n_j \psi^{j0\nu} \to 0$$

for surface $S \to \infty$. So adding $\partial_\rho \psi^{\rho\mu\nu}$ keeps all the good properties.

Improving $T^{\mu\nu}$

So consider $\psi^{\rho\mu\nu} = A^\nu F^{\mu\rho}/4\pi$, and adding

$$\frac{1}{4\pi}\partial_\rho\left(A^\nu F^{\mu\rho}\right) = \frac{1}{4\pi}\left(\partial_\rho A^\nu\right)F^{\mu\rho}$$

because $\partial_\rho F^{\mu\rho} = 0$ in the absence of a source J^μ. But this is just what we need to add to $T^{\mu\nu}$ to make

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{4\pi}F^{\mu\rho}\partial_\rho A^\nu = -\frac{1}{4\pi}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right).$$

This expression has all the good properties and is also gauge invariant and symmetric. Furthermore,

$$\Theta^{0i} = -\frac{1}{4\pi}F^{0j}F^{i}{}_{j} = \frac{1}{4\pi}E_j\epsilon_{ijk}B_k = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i,$$

the correct momentum density or energy flux, as given by Poynting.

Ambiguities in $\mathcal{L}$

$-\dfrac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$ is gauge invariant, but the interaction term in the action, $-\dfrac{1}{c}\displaystyle\int d^4x J_\mu(x^\rho)A^\mu(x^\rho)$ is not, so $\mathcal{L}$ is not a unique function of the physical state ($\vec{E}$ and $\vec{B}$ and J^μ). Is there an ambiguity in the action under a gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu\Lambda$? This adds a piece to the action $\Delta A = -(1/c)\int d^4x J^\mu \partial_\mu \Lambda$. But

$$\int d^4x J^\mu \partial_\mu \Lambda = \underbrace{\int_S n_\mu\, J^\mu\, \Lambda}_{\xrightarrow[S\to\infty]{} 0} - \int d^4x \Lambda \underbrace{\partial_\mu J^\mu}_{0},$$

so this will not affect the action.

More generally, adding a total divergence to the lagrangian density in a field theory, like adding a total time derivative in a particle theory, does not affect the equations of motion, and is irrelevant to the physics.

$\Theta^{\mu\nu}$ with currents

The energy-momentum tensor of the electromagnetic field is

$$\Theta^{\mu\nu}_{\text{EM}} = -\frac{1}{4\pi}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right),$$

and is conserved ($\partial_\mu \Theta^{\mu\nu}_{\text{EM}} = 0$ if there are no sources. What if there are?

$$\begin{aligned}
4\pi\partial_\mu\Theta^{\mu\nu} &= \partial_\mu\left(F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \\
&= \left(\partial_\mu F^{\mu\rho}\right)F_{\rho}{}^{\nu} + F^{\mu\rho}\partial_\mu F_{\rho}{}^{\nu} + \frac{1}{2}F^{\alpha\beta}\partial^\nu F_{\alpha\beta} \\
&= \frac{4\pi}{c}J^\rho F_{\rho}{}^{\nu} + \frac{1}{2}F^{\alpha\beta}\left(\partial_\alpha F_{\beta}{}^{\nu} - \partial_\beta F_{\alpha}{}^{\nu} + \partial^\nu F_{\alpha\beta}\right) \\
&= \frac{4\pi}{c}J^\rho F_{\rho}{}^{\nu} + \frac{1}{2}F^{\alpha\beta}\eta^{\nu\rho}\underbrace{\left(\partial_\alpha F_{\beta\rho} + \partial_\beta F_{\rho\alpha} + \partial_\rho F_{\alpha\beta}\right)}_{=0 \text{ as } \mathbf{d}F = ddA = 0} \\
&= \frac{4\pi}{c}J^\rho F_{\rho}{}^{\nu} \neq 0.
\end{aligned}$$

Not conserved!

$P^\nu_{\rm EM}$ is not conserved

Thus the total 4-momentum of the electromagnetic field

$$P^\nu_{\rm EM} = \frac{1}{c}\int d^3x \Theta^{0\nu}(\vec{x}),$$

is not conserved, but rather

$$\begin{aligned}
\frac{dP^\nu_{\rm EM}}{dt} &= \frac{1}{c}\frac{d}{dt}\int d^3x\, \Theta^{0\nu}(\vec{x}) = \int d^3x\, \partial_0\Theta^{0\nu}(\vec{x}) \\
&= \frac{1}{c}\int d^3x J^\rho(\vec{x}) F_\rho{}^\nu(\vec{x}) - \frac{1}{c}\int d^3x \partial_i \Theta^{i\nu} \\
&= \frac{1}{c}\int d^3x J^\rho(\vec{x}) F_\rho{}^\nu(\vec{x}),
\end{aligned}$$

as the second term is the integral of a divergence.

Total Momentum is conserved

Consider a charged particle of mass m_i, charge q_i at point $\vec{x}_i(t)$. Its mechanical 4-momentum changes by

$$\frac{dP^\nu_{(i)}}{dt} = \frac{1}{\gamma_i}\frac{dP^\nu_{(i)}}{d\tau} = \frac{1}{\gamma_i}\frac{q_i}{c}F^\nu{}_\rho(\vec{x}_i)U^\rho_i.$$

This particle corresponds to a 4-current

$$\begin{aligned} J^\rho &= (c\rho, \vec{J}) = (cq_i\delta^3(\vec{x}-\vec{x}_i), q_iu_i\delta^3(\vec{x}-\vec{x}_i) \\ &= q_i\gamma_i^{-1}U^\rho_i\delta^3(\vec{x}-\vec{x}_i). \end{aligned}$$

Plugging this into our expression for the change in the momentum of the electromagnetic field, we have

$$\frac{dP^\nu_{\text{EM}}}{dt} = \frac{q_i}{c}\int d^3x F_\rho{}^\nu(\vec{x})\gamma_i^{-1}U^\rho_i\delta^3(\vec{x}-\vec{x}_i) = -\frac{q_i}{c\gamma_i}F^\nu{}_\rho(\vec{x}_i)U^\rho_i,$$

and the *total momentum*, $P^\nu_{\text{EM}} + P^\nu_{(i)}$ is conserved.

Equations of Motion for A^μ

Euler-Lagrange tell us

$$\partial_\sigma F^{\sigma\mu} = \partial_\sigma \partial^\sigma A^\mu - \partial^\mu \partial_\sigma A^\sigma = \frac{4\pi}{c} J^\mu .$$

If we knew $\partial_\sigma A^\sigma = 0$ (the Lorenz condition), we could discard second term, and have

$$\partial_\sigma \partial^\sigma A^\mu = \frac{4\pi}{c} J^\mu ,$$

which has solutions given by
1) a particular solution, given in terms of the Green's function on J, and
2) an arbitrary solution of the homogeneous wave equation $\partial_\sigma \partial^\sigma A^\mu = 0$. The homogeneous solution is

$$\sum_{\vec{k}} \left(A^\mu_{\vec{k}+} e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} t} + A^\mu_{\vec{k}-} e^{i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t} \right) ,$$

where $\omega = c|\vec{k}|$, where $\omega = c|\vec{k}|$.

But we assumed the Lorenz condition, which constrains the coefficients $\omega A^0_{\vec{k}\pm} \mp \vec{k} \cdot \vec{A}_{\vec{k}\pm} = 0$. These are the solutions for an electromagnetic wave in empty space.

Without imposing the Lorenz condition,

$$\partial_\sigma \partial^\sigma A^\mu - \partial^\mu \partial_\sigma A^\sigma = 0,$$

which is inadequate to determine the evolution of $A^\mu(\vec{x}, t)$ in time. Fourier transform:
$k_\sigma k^\sigma \tilde{A}^\mu(k^\nu) - k^\mu k_\sigma \tilde{A}^\sigma(k^\nu) = 0$, which is not four independent equations, because dotting with k_ρ gives

$$k_\sigma k^\sigma k_\mu \tilde{A}^\mu(k^\nu) - k_\mu k^\mu k_\sigma \tilde{A}^\sigma(k^\nu) = (k^2 - k^2) k_\rho \tilde{A}^\rho(k^\nu) = 0,$$

telling us nothing about $\tilde{A}^\rho(k^\nu)$. Euler-Lagrange only determine the components transverse to k.
This is gauge invariance again. No physics constrains the gauge transformation $\Lambda(\vec{x}, t)$ in the future, so A^μ is underdetermined.

Solving the inhomogeneous equation

But we are free to impose the Lorenz condition. Let's do so.

Now we turn to the inhomogeneous equation

$$\Box A^\mu = \partial_\beta \partial^\beta A^\mu = \frac{4\pi}{c} J^\mu,$$

with the solution

$$A^\mu(x) = \frac{4\pi}{c} \int d^4x' D(x, x') J^\mu(x'),$$

where $D(x, x')$ is a Green's function for D'Alembert's equation

$$\Box_x D(x, x') = \delta^4(x - x').$$

We are interested in solving this in all of spacetime. No boundaries, translation invariance, so $D(x, x') = D(x - x') = D(z)$. Solve by Fourier transform:

write $$D(z) = \frac{1}{(2\pi)^4} \int d^4k^\mu \tilde{D}(k^\mu) e^{-ik_\mu z^\mu}.$$

Fourier transformed equation

As $\delta^4(z^\mu) = \frac{1}{(2\pi)^4}\int d^4k e^{-ik_\mu z^\mu}$, we have $k^2\tilde{D}(k^\mu) = -1$, so the solution for the Green's function is

$$\tilde{D}(k^\mu) = -\frac{1}{k^2}, \quad \text{and} \quad D(z^\mu) = -\frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik_\mu z^\mu}}{k^2}.$$

As $\delta^4(z^\mu) = \frac{1}{(2\pi)^4}\int d^4k e^{-ik_\mu z^\mu}$, the solution for the Green's function is

$$\tilde{D}(k^\mu) = -\frac{1}{k^2}, \quad \text{and} \quad D(z^\mu) = -\frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik_\mu z^\mu}}{k^2}.$$

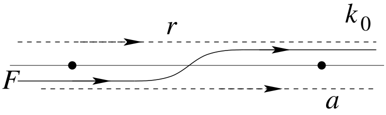
Looks like what we did for Poisson, but here $k^2 = 0$ is more difficult, as it requires only $k_0^2 = \vec{k}^{\,2}$, not $\vec{k} = 0$.
For Poisson, trouble from $\vec{k} = 0$ gives ambiguity of $\psi = V_0 + \vec{r}\cdot\vec{C}$, a uniform $\vec{E}$ and ambiguous constant in Φ.
For wave equation: arbitrary waves satisfying free wave equation.
Deform the integration path to resolve the singularities.

Disambiguate with Contour choice

Clarify ambiguity by specifying how to avoid the singularities and writing

$$D(z) = -\frac{1}{(2\pi)^4}\int d^3k e^{i\vec{k}\cdot\vec{z}} \int_\Gamma dk_0 \frac{e^{-ik_0 z^0}}{k_0^2 - |\vec{k}|^2}.$$

Specifying contour Γ's avoidance of the poles at $k_0 = \pm|\vec{k}|$. Three such contours are shown. Integrand analytic except at poles so the contours may be deformed while avoiding the poles.



The retarded (r), advanced (a), and Feynman (F) contours for defining the Green's function.

Retarded Green's function

Consider D for contour r. If source at $x' = 0$, and we look for A later, $z^0 > 0$. Close contour in lower half plane, where $\left|e^{-ik_0 z^0}\right| = e^{-|\mathrm{Im}\, k_0| z^0} \underset{|k|\to\infty}{\longrightarrow} 0$, so this semicircle adds nothing. So $D = -2\pi i$ times the sum of the residues. minus because clockwise. The residues are

$$\operatorname*{Res}_{k_0=|\vec{k}|} \frac{e^{-ik_0 z^0}}{(k_0+|\vec{k}|)(k_0-|\vec{k}|)} + \operatorname*{Res}_{k_0=-|\vec{k}|} \frac{e^{-ik_0 z^0}}{(k_0+|\vec{k}|)(k_0-|\vec{k}|)}$$

$$= \frac{e^{-i|\vec{k}|z^0}}{2|\vec{k}|} + \frac{e^{i|\vec{k}|z^0}}{-2|\vec{k}|} = -i\frac{\sin(|\vec{k}|z^0)}{|\vec{k}|}.$$

But if $z^0 < 0$ close in upper half plane, no residues, $D = 0$, so all together

$$D_r(z) = \frac{\Theta(z^0)}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{z}} \frac{\sin(|\vec{k}|z^0)}{|\vec{k}|}.$$

D is rotationally invariant, so we may choose the North pole along $\vec{z}$ using spherical coordinates.

We get

$$D_r(z) = \frac{\Theta(z^0)}{(2\pi)^2}\int_0^\infty k^2\,dk\,d\theta\sin\theta\,e^{ikR\cos\theta}\,\frac{\sin(kz^0)}{k}$$
$$= \frac{\Theta(z^0)}{2\pi^2 R}\int_0^\infty dk\sin(kR)\sin(kz^0),$$

where $R = |\vec{z}|$.
This is the *retarded Green's function* aka *causal*, as the effects on A^μ of the source are felt only after the source acts.

The contour a gives the *advanced Green's function* useful only if you want to configure an **incoming** field which would magically be totally dissolved by a given source.

Finally the contour F gives the Feynman propagator, which is used in quantum field theory.

Simplifying D

We may simplify D_r by noting

$$
\begin{aligned}
\sin(kR)\sin(kz^0) &= \frac{1}{2}\left[\cos(k(R-z^0)) - \cos(k(R+z_0))\right] \\
&= \frac{1}{4}\Big[e^{i(z_0-R)k} - e^{i(z_0+R)k} + e^{i(z_0-R)(-k)} \\
&\qquad +e^{i(z_0-R)(-k)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\text{so} \quad D_r(z) &= \frac{\Theta(z^0)}{8\pi^2 R}\int_{-\infty}^{\infty} dk\left[e^{i(z_0-R)k} - e^{i(z_0+R)k}\right] \\
&= \frac{\Theta(z^0)}{4\pi R}\left[\delta(z_0-R) - \delta(z_0+R)\right] \\
&= \frac{\Theta(z^0)}{4\pi R}\delta(z_0-R),
\end{aligned}
$$

where the second δ was dropped because both z^0 and R are positive. So the Green's function only contributes when the source and effect are separated by a lightlike path, with $\Delta z^0 = |\Delta\vec{z}|$.

Full solution for A^μ

So how do we describe the field when we know what the sources are throughout space-time? We can use any of the Green's functions to get the inhomogeneous contribution, and then allow for an arbitrary solution of the homogeneous equation. Thus we can write

$$\begin{aligned} A^\mu &= A^\mu_{\text{in}}(x) + \frac{4\pi}{c}\int d^4x' D_r(x-x')J^\mu(x') \\ &= A^\mu_{\text{out}}(x) + \frac{4\pi}{c}\int d^4x' D_a(x-x')J^\mu(x'). \end{aligned}$$

If the sources are confined to some finite region of space-time, there will be no contribution from D_r at times earlier than the first source, and $A^\mu_{\text{in}}(x)$ describes the fields before that time. Also after the last time that the source influences things, the field will be given by $A^\mu_{\text{out}}(x)$ alone.

Radiation Field.

Of course the source may be persistent, for example if there is a net charge, but we may often consider that the effect of the source is confined to the change from $A^\mu_{\text{in}}(x)$ to $A^\mu_{\text{out}}(x)$. Then we define the radiation field to be

$$A^\mu_{\text{rad}}(x) = A^\mu_{\text{out}}(x) - A^\mu_{\text{in}}(x) = \frac{4\pi}{c}\int d^4x' D(x-x')J^\mu(x'),$$

where $D(z) := D_r(z) - D_a(z)$.

Better expression for J^μ from charges

The expression we wrote earlier for the current density of a point charge,

$$J^\rho = q_i \gamma_i^{-1} U_i^\rho \delta^3(\vec{x} - \vec{x}_i)$$

can be written in this four-dimensional language as

$$\begin{aligned} J^\rho(x^\mu) &= q_i \int dt \delta(t - x^0/c) \gamma_i^{-1} U_i^\rho \delta^3(\vec{x} - \vec{x}_i(t)) \\ &= q_i c \int d\tau \delta^4(x^\mu - x_i^\mu(\tau)) U_i^\rho, \end{aligned}$$

where τ measures proper time along the path of the particle.