Lecture 14 March 22, 2010

Magnetic Moment and Spin

Current loops have magnetic moments. Rotating charge distributions as well.

$$\vec{m} = \frac{1}{2}\int \vec{r} \times \vec{J}(\vec{r}) d^3r$$

for moving charges $\vec{m} = \frac{1}{2}\int \rho_Q \vec{r} \times \vec{v}(\vec{r}) d^3r$ looks like the angular momentum with $\rho_m \to \rho_Q$. If ratio is uniform,

$$\frac{\rho_Q}{\rho_m} = \frac{Q}{m} \Longrightarrow \vec{m} = \frac{1}{2}\frac{Q}{m}\vec{L} \text{ (SI units) } = \frac{Q}{2mc}\vec{L} \text{ (Gaussian).}$$

If ratio is not constant, there is a correction factor g, with

$$\vec{m} = g\frac{Q}{2mc}\vec{L}.$$

For an electron in an atom, angular momentum is sum of orbital $\vec{L}$ and spin $\vec{s}$.

spin (rotation) $s_z = \pm\hbar/2$ For atomic-sized particles we call the magnetic moment $\vec{\mu}$ instead of $\vec{m}$.

For orbital motion, we certainly have $\rho_Q/\rho_m = e/m$ constant, so we expect $g = 1, \vec{\mu} = e\vec{L}/2mc$. For spin, no classical picture, but Dirac equation gives a different value, $g = 2$.

The magnetic moment can be measured in a constant magnetic field, as $U = -\vec{\mu}\cdot\vec{B}$. Particles have quantized angular momenta $\hbar\ell$, with ℓ an integer, and the component in the direction of the magnetic field (say z) is restricted to $L_z = m\hbar$, m an integer $\in [-\ell, \ell]$, with an odd number of possible values. The split levels is known as the *Zeeman effect*.

Before spin, expect an odd number, *normal Zeeman*, and $g = 1$

with spin, *anomolous Zeeman*, can have even number, and $g \neq 1$.

A magnetic moment in a uniform $\vec{B}$ has energy $U = -\vec{\mu}\cdot\vec{B}$ and torque $\tau = \vec{\mu}\times\vec{B}$. For a magnetic dipole at rest with respect to observer $\mathcal{O}'$ with angular momentum = spin $\vec{s}\,'$, with magnetic moment $\vec{\mu} = \frac{ge}{2mc}\vec{s}\,'$, we have

$$\frac{d\vec{s}\,'}{dt'} = \frac{ge}{2mc}\vec{s}\,'\times\vec{B}\,', \qquad U' = -\frac{ge}{2mc}\vec{s}\,'\cdot\vec{B}\,'. \qquad (1)$$

The Lorentz transformation $\mathcal{O}\to\mathcal{O}'$ is, to first order in $\vec{v}/c$,

$$\begin{aligned} A^\mu{}_\nu &= \delta^\mu_\nu - v^\mu\delta^0_\nu/c + v_\nu\delta^\mu_0/c \quad \text{so} \\ F'^{\,ij} &= F^{ij} - v^iF^{0j}/c + v_jF^{i0}/c \\ &\Longrightarrow -\epsilon_{ij\ell}B'_\ell = -\epsilon_{ij\ell}B_\ell + 2v^iE_j/c, \end{aligned}$$

$$\text{or} \quad \vec{B}\,' = \vec{B} - \frac{\vec{v}}{c}\times\vec{E} \qquad \text{to order } \mathcal{O}(v^2/c^2).$$

Suppose $\mathcal{O}$ = atom, $\mathcal{O}'$ = electron with potential **energy** $V(r)$, so

$$e\vec{E} = -\vec{\nabla}V(\vec{r}) = -\frac{\vec{r}}{r}\frac{dV}{dr},$$

Then Eq. 1: $\frac{d\vec{s}\,'}{dt'} = \frac{ge}{2mc}\vec{s}\,' \times \vec{B}\,', \qquad U' = -\frac{ge}{2mc}\vec{s}\,' \cdot \vec{B}\,'$

seems to give

$$\begin{aligned}
U' &= -\vec{\mu} \cdot \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right) \\
&= -\frac{ge}{2mc}\vec{s}\,' \cdot \vec{B} - \frac{g}{2mc^2}\vec{s}\,' \cdot (\vec{v} \times \vec{r})\,\frac{1}{r}\frac{dV}{dr} \\
&= -\frac{ge}{2mc}\vec{s}\,' \cdot \vec{B} + \frac{g}{2m^2c^2}\vec{s}\,' \cdot \vec{L}\frac{1}{r}\frac{dV}{dr}, \qquad (2) \\
\frac{d\vec{s}\,'}{dt'} &= \frac{ge}{2mc}\vec{s}\,' \times \vec{B} - \frac{g}{2m^2c^2}\vec{s}\,' \times \vec{L}\,\frac{1}{r}\frac{dV}{dr} \qquad (3)
\end{aligned}$$

But this is not verified experimentally!

Energy includes

- dipole energy, $\frac{ge}{2mc}\vec{s}\,' \times \vec{B}$, gives correct anomalous Zeeman effect with the expected (from Dirac) $g = 2$, and
- spin-orbit, fine-structure splitting,

$$-\frac{g}{2m^2c^2}\vec{s}\,' \times \vec{L}\,\frac{1}{r}\frac{dV}{dr}$$

better fit to experiment with $g = 1$!

What did we do wrong?
We will discuss this two ways
1) We will examine more carefully the lorentz transformations from $\mathcal{O}$ to $\mathcal{O}'$, and do better at asking who $\mathcal{O}'$ is.
2) We will figure out how to express these things co-variantly, so we don't need to worry about doing LT's.

First, fixing our Lorentz transformations.

Two boosts contain a rotation

1) What did we do here?
a) We have a $\vec{B}$ and $\vec{E}$ in the "lab", the rest frame of the atom, and we transformed to the frame $\mathcal{O}'$ with respect to whom the electron was at rest (momentarily) at time t, with a pure boost.
b) $\mathcal{O}'$ calculated how much $\vec{s}\,'$ would change in time Δt.
c) we transformed back to the lab. (this may not have been obvious, because to first order in v we don't expect U and $\vec{s}$ to change.) We used a pure boost.
But $\vec{v}$ does change in the time Δt! So the combined boosts were actually $A^\mu{}_\nu(-\vec{v}-\Delta\vec{v})A^\nu{}_\rho(\vec{v})$. That is,

$$e^{-(\vec{v}+\Delta\vec{v})\cdot\vec{K}}e^{\vec{v}\cdot\vec{K}} = 1\!\!1 - \Delta\vec{v}\cdot\vec{K} + \frac{1}{2}v_i(\Delta v)_j[K_i,K_j] + \mathcal{O}(v^2\Delta v)$$

$$= 1\!\!1 - \Delta\vec{v}\cdot\vec{K} - \frac{1}{2}\left(\vec{v}\times\Delta\vec{v}\right)\cdot\vec{S}.$$

The last term means there is an apparent rotation in the lab even if the electron did not rotate in its rest frame!

Thomas precession

There is a Thomas precession

$$\vec{\omega}_T = -\frac{1}{2}\vec{v}\times\vec{a} = -\frac{1}{2m}\vec{v}\times\vec{F} = -\frac{1}{2m}\vec{r}\times\vec{v}\frac{1}{r}\frac{dV}{dr}.$$

Adding $\vec{\omega}_T \times \vec{s}\,'$ into the equation for $\dfrac{d\vec{s}\,'}{dt}$ we have the corrected formula

$$\frac{d\vec{s}\,'}{dt'} = \vec{s}\,' \times \left(\frac{ge}{2mc}\vec{B} - \frac{g-1}{2m^2c^2}\,\vec{L}\,\frac{1}{r}\frac{dV}{dr}\right).$$

This same change needs to be made in the energy expression[1], changing the g in the fine structure term to $g-1$. As g is very nearly 2, $g-1$ is very nearly not there.

This is a good physical argument, but perhaps you are worried about signs? Let's be more formal.

[1] why? $\tau = -i\vec{L}U = \mu\times\vec{C} \Longrightarrow U = -\mu\cdot\vec{C}$. See lecture notes.

Covariant treatment of spin

Spin is angular momentum, $s_i \sim \epsilon_{ijk}\mathcal{L}_{jk}$, a 3-vector which is the angular momentum *in the rest frame of the particle.* So we can use $U^\alpha = (c, 0, 0, 0)$ in that frame, to define $S^\alpha \sim \epsilon^{\alpha\beta\gamma\xi}U_\beta\mathcal{L}_{\gamma\xi}$.
S^α is a 4-vector but $U_\alpha S^\alpha = 0$, only three independent components, all spatial in the rest frame of the particle. In the rest frame, the first of Eq. (1) is

$$\frac{dS^i}{d\tau} = \frac{ge}{2mc}\epsilon_{ijk}S^j(-\epsilon_{k\ell p}F^{\ell p}/2) = (ge/2mc)S^\ell F^i{}_\ell.$$

Can we simply replace i with α to get a 4-D equation? No, we only know it to be true in the rest frame, it doesn't determine $U_\alpha dS^\alpha/d\tau$. But we may write

$$\frac{dS^\alpha}{d\tau} - \frac{1}{c^2}U^\alpha U_\beta\frac{dS^\beta}{d\tau} = \frac{ge}{2mc}\left(F^\alpha{}_\beta S^\beta - \frac{1}{c^2}U^\alpha U_\zeta F^\zeta{}_\beta S^\beta\right),$$

As $U_\alpha S^\alpha = 0$ at all times, $U_\alpha dS^\alpha/d\tau = -(dU_\alpha/d\tau)S^\alpha$. If the only force on the particle is the Lorentz force[2]

$$\frac{dU_\alpha}{d\tau} = \frac{1}{m}\frac{dp_\alpha}{d\tau} = \frac{1}{m}\frac{e}{c}F_{\alpha\beta}U^\beta.$$

So

$$\begin{aligned}\frac{dS^\alpha}{d\tau} &= \frac{ge}{2mc}\left(F^\alpha{}_\beta S^\beta - \frac{1}{c^2}U^\alpha U_\zeta F^\zeta{}_\beta S^\beta\right) - \frac{e}{mc^3}U^\alpha S^\beta F_{\beta\gamma}U^\gamma \\ &= \frac{e}{mc}\left[\frac{g}{2}F^\alpha{}_\beta S^\beta + \frac{1}{c^2}\left(\frac{g}{2}-1\right)U^\alpha S_\beta F^{\beta\zeta}U_\zeta\right] \qquad (4)\end{aligned}$$

[2] We assume so, with $\vec{B}$ uniform. See lecture notes for complications.

In rest frame $S'^{\mu} = (0, \vec{s})$, so applying a Lorentz transformation

$$S^0 = \gamma \vec{\beta} \cdot \vec{s}, \qquad \vec{S} = \vec{s} + \frac{\gamma^2}{\gamma + 1} (\vec{\beta} \cdot \vec{s}) \vec{\beta}.$$

[To see this, first take $\vec{v} \parallel x$, then replace $s_x \to \vec{\beta} \cdot \vec{s}/\beta$]. To compare with previous result, which is $\mathcal{O}(v^1)$, care is needed, because $d\vec{\beta}/d\tau = e\vec{E}/mc$ is zeroth order in v. So

$$\frac{d\vec{s}}{d\tau} = \frac{d\vec{S}}{d\tau} - \frac{e}{2mc^2} \left[(\vec{s} \cdot \vec{E}) \vec{v} + (\vec{s} \cdot \vec{v}) \vec{E} \right],$$

and using (4),

$$\begin{aligned} \frac{dS_i}{d\tau} &= \frac{e}{mc} \left[\frac{g}{2} F^i{}_j s_j + \frac{g}{2} F^i{}_0 \vec{\beta} \cdot \vec{s} - \frac{1}{c^2} \left(\frac{g}{2} - 1 \right) v_i s_j F^{j0} c \right] \\ \frac{d\vec{S}}{d\tau} &= \frac{e}{mc} \left[\frac{g}{2} \vec{s} \times \vec{B} + \frac{g}{2} (\vec{v} \cdot \vec{s}) \vec{E}/c - \frac{1}{c} \left(\frac{g}{2} - 1 \right) \vec{v} (\vec{s} \cdot \vec{E}) \right]. \end{aligned}$$

To repeat:

$$\frac{d\vec{s}}{d\tau} = \frac{d\vec{S}}{d\tau} - \frac{e}{2mc^2}\left[(\vec{s}\cdot\vec{E})\vec{v} + (\vec{s}\cdot\vec{v})\vec{E}\right] \qquad \text{to } \mathcal{O}(v),$$

$$\frac{d\vec{S}}{d\tau} = \frac{e}{mc}\left[\frac{g}{2}\vec{s}\times\vec{B} + \frac{g}{2}(\vec{v}\cdot\vec{s})\vec{E}/c - \frac{1}{c}\left(\frac{g}{2}-1\right)\vec{v}(\vec{s}\cdot\vec{E})\right].$$

Putting the terms together, to first order in v,

$$\begin{aligned}\frac{d\vec{s}}{d\tau} &= \frac{e}{mc}\left[\frac{g}{2}\vec{s}\times\vec{B} + \frac{g}{2}(\vec{v}\cdot\vec{s})\vec{E}/c - \frac{1}{c}\left(\frac{g}{2}-1\right)\vec{v}(\vec{s}\cdot\vec{E})\right] \\ &\qquad - \frac{e}{2mc^2}\left[(\vec{s}\cdot\vec{E})\vec{v} + (\vec{s}\cdot\vec{v})\vec{E}\right] \\ &= \frac{e}{mc}\vec{s}\times\left[\frac{g}{2}\vec{B} - \frac{g-1}{2c}\vec{v}\times\vec{E}\right]. \end{aligned} \tag{5}$$

Notice the $\vec{E}$ term has $g-1$

$g \approx 2$

The Dirac equation predicts $g = 2$ for the electron. This is not the full story, because there are small corrections from Quantum Field Theory.

For $g = 2$, the second term in

$$\frac{d\vec{s}}{d\tau} = \frac{d\vec{S}}{d\tau} - \frac{e}{2mc^2}\left[(\vec{s}\cdot\vec{E})\vec{v} + (\vec{s}\cdot\vec{v})\vec{E}\right],$$

vanishes. If we have a pure magnetic field (with ${F^0}_\beta = 0$), the zero'th component vanishes as well, so $S^0 = \gamma\vec{\beta}\cdot\vec{s}$ is a constant. But so are γ and $|\beta|$, so the helicity $\hat{\beta}\cdot\vec{s}$ is conserved.
Thus the precession of helicity is a very sensitive way to measure $g - 2$.

Test of Quantum Field Theory

In quantum field theory there are many higher order (in $e^2/4\pi\hbar c \approx 1/137$) which give corrections. Checking the measured values of $g-2$ against the (extremely complicated) theoretical calculations is the finest check on theoretical ideas in science.

Experiment says[3]

$$\frac{g-2}{2} = \begin{array}{r} 0.001\,159\,652\,180\,73 \\ \pm 0.000\,000\,000\,000\,28 \end{array}$$

and theory says[4]

$$\frac{g-2}{2} = \begin{array}{r} 0.001\,159\,652\,182\,79 \\ \pm 0.000\,000\,000\,007\,71 \end{array}$$

certainly one of the most accurately measured quantities in physics.

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[3] Hanneke, Forgwell, Gabrielse, Phys. Rev. Lett. 100, 120801 (2008). 34.

[4] Aoyama, Hayakawa, Kinoshita, Nio, Phys. Rev. D 77, 053012 (2008)