Note on Representations, the Adjoint rep, the Killing form, and antisymmetry of ${c_{ij}}^k$

A representation of a group, to a physicist[1], is a vector space $\vec{V} = \sum V^i \hat{e}_i$ on which elements g of a group G can act as a linear transformation

$$g : \vec{V} \to \vec{V}\,' \quad \text{with } V_i' = \sum_j M_{ij}(g) V_j,$$

with the requirement that the group multiplication is preserved,

$$g_1(g_2(V)) = (g_1 g_2)(V),$$

which requires

$$M_{ik}(g_1 g_2) = \sum_j M_{ij}(g_1) M_{jk}(g_2).$$

Notice that the active interpretation of the group acting on components can be reinterpreted passively, as taking the basis elements

$$g : \hat{e}_i \to \sum_i M_{ij}(g) \hat{e}_i,$$

where the contraction is on the first index rather than the second.

For a Lie group the Lie algebra of infinitesimal generators is itself a representation, called the **adjoint representation** with $\vec{A} = \sum A^b L_b$ transforming under the group by

$$g : \vec{A} \to \vec{A}\,' = g \vec{A} g^{-1}. \tag{1}$$

The need for g's on both sides is to keep $\vec{A}\,'$ a member of the Lie algebra, which can be though of as derivatives of group transformations about the identity,

$$L_b = -i \frac{\partial}{\partial \theta^b} e^{i\theta^{(b)} L_b},$$

[1]To a physicist, the vector space acted upon is called the representation. This is often a space of fields or coordinates. To mathematicians the representation consists of the matrices M_{ij}, or more accurately the mapping from elements of the group into matrices, $g \to M(g)$. Mathematicians have no interest in what is being acted on.

and (1) follows from the fact that the group itself can be transformed by group elements by similarity,

$$g_1 : g_2 \rightarrow g_1 g_2 g_1^{-1},$$

which obeys the requirement $g_1(g_2(g_3)) = (g_1 g_2)(g_3)$. Notice that this would not be true if we interchanged the g and g^{-1} in (1).

A representation of a Lie group generates an associated representation of the Lie algebra, with

$$M_{ij}(L_b) = -i \frac{\partial}{\partial \theta^b} M_{ij}(e^{i\theta^b L_b}),$$

so differentiating the adjoint representation for a transformation by $g = expi\theta^c L_c$:

$$\vec{A}' = g\vec{A}g^{-1} = \sum M^{\mathrm{adj}}_{ab}(g) A^b L_a$$

with respect to θ_c and setting θ to zero gives

$$\begin{aligned} [L_c, A^b L_b] &= \sum M^{\mathrm{adj}}_{ab}(L_c) A^b L_a \\ &= i{c_{cb}}^a A^b L_a, \end{aligned}$$

from which we can derive

$$M^{\mathrm{adj}}_{ab}(L_c) = i{c_{cb}}^a.$$

This agrees with the definition of the adjoint representation in Schensted[2] but disagrees with the representation Γ below which Georgi[3] calls the adjoint representation. In fact, the two are adjoints in another sense: on the group, $M(g) = \Gamma^T(g^{-1})$, and it is easy to show that any representation has a dual in this sense. Once we normalize our generators, as discussed below, so that the killing form is a multiple of the identity and the structure constants are totally antisymmetric, the two become equal anyway, so I proceed with Georgi's, which I used first.

The adjoint representation is defined in terms of the structure constants by

$$\Gamma^{\mathrm{adj}}_{jk}(L_i) = i{c_{ji}}^k,$$

[2]Irene Verona Schensted, "A Course on the Application of Group Theory to Quantum Mechanics", Neo Press, Maine, 1976, P. 283.

[3]Howard Georgi, "Lie Algebras in Particle Physics", Addison-Wesley, 1982.

with the structure constants defined by

$$[L_i, L_j] = i c_{ij}{}^k L_k.$$

We can check that this is a representation by

$$\begin{aligned}
\Gamma^{\mathrm{adj}}_{ab}([L_i, L_j]) &= i c_{ij}{}^k \Gamma^{\mathrm{adj}}_{ab}(L_k) = -c_{ij}{}^k c_{ak}{}^b \\
&= c_{ai}{}^k c_{jk}{}^b + c_{ja}{}^k c_{ik}{}^b \qquad \text{(Jacobi identity)} \\
&= \Gamma^{\mathrm{adj}}_{ak}(L_i)\Gamma^{\mathrm{adj}}_{kb}(L_j) - \Gamma^{\mathrm{adj}}_{ak}(L_j)\Gamma^{\mathrm{adj}}_{kb}(L_i) \\
&= \left[\Gamma^{\mathrm{adj}}(L_i), \Gamma^{\mathrm{adj}}(L_j)\right]_{ab}
\end{aligned}$$

The Killing form $\beta : \mathcal{L} \times \mathcal{L} \mapsto \mathbb{R}$ is given by

$$\beta(L_i, L_j) = \mathrm{Tr}\left(\Gamma^{\mathrm{adj}}(L_i)\Gamma^{\mathrm{adj}}(L_j)\right) = -c_{ai}{}^b c_{bj}{}^a.$$

Note that $\beta(L_i.L_j)$ is a real symmetric matrix, which can be diagonalized by an orthogonal matrix, which corresponds to a change in basis of the L_i's. Furthermore, the scale of each new L_i can be adjusted to make the diagonal elements 0, 1, or -1, but as only real scalings are allowed, and their squares enter, the signs cannot be undone. It can be shown that for semisimple compact groups the signs are positive. For historical reasons we always normalize such groups so that $\beta(L_i, L_j) = 2\delta_{ij}$. Assume this is done.

Consider the trace in the adjoint representation

$$\begin{aligned}
&\mathrm{Tr}\left\{\left[\Gamma^{\mathrm{adj}}(L_i), \Gamma^{\mathrm{adj}}(L_j)\right]\Gamma^{\mathrm{adj}}(L_k)\right\} = i c_{ij}{}^\ell\, \mathrm{Tr}\left\{\Gamma^{\mathrm{adj}}(L_\ell)\Gamma^{\mathrm{adj}}(L_k)\right\} = 2 i c_{ij}{}^k \\
&\quad = \mathrm{Tr}\left\{\Gamma^{\mathrm{adj}}(L_i)\Gamma^{\mathrm{adj}}(L_j)\Gamma^{\mathrm{adj}}(L_k)\right\} - \mathrm{Tr}\left\{\Gamma^{\mathrm{adj}}(L_j)\Gamma^{\mathrm{adj}}(L_i)\Gamma^{\mathrm{adj}}(L_k)\right\} \\
&\quad = \mathrm{Tr}\left\{\Gamma^{\mathrm{adj}}(L_k)\Gamma^{\mathrm{adj}}(L_i)\Gamma^{\mathrm{adj}}(L_j)\right\} - \mathrm{Tr}\left\{\Gamma^{\mathrm{adj}}(L_i)\Gamma^{\mathrm{adj}}(L_k)\Gamma^{\mathrm{adj}}(L_j)\right\} \\
&\quad = \mathrm{Tr}\left\{\left[\Gamma^{\mathrm{adj}}(L_k), \Gamma^{\mathrm{adj}}(L_i)\right]\Gamma^{\mathrm{adj}}(L_j)\right\} = \mathrm{Tr}\left\{\Gamma^{\mathrm{adj}}([L_k, L_i]\,\Gamma^{\mathrm{adj}}(L_j)\right\} = 2 i c_{ki}{}^j \\
&= -2 i c_{ik}{}^j
\end{aligned}$$

which shows that $c_{ij}{}^k$, which we already know is antisymmetric under interchange of its first two indices, is also antisymmetric under interchanges with the third.