Physics 504 Project #2 Solutions
Due: Thursday, February 28, 2009

A We are asked to find the electrostatic potential outside a charged conductor in the shape of an oblate ellipsoid,

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{A}\right)^2 + \left(\frac{z}{B}\right)^2 = 1,$$

with $A > B$. This is an oblate ellipsoid. So we should look at oblate ellipsoidal coordinates,

$$\begin{aligned} x &= a\,\cosh\mu\,\cos\nu\,\cos\phi \\ y &= a\,\cosh\mu\,\cos\nu\,\sin\phi \\ z &= a\,\sinh\mu\,\sin\nu. \end{aligned}$$

Define $\xi = \sin\nu \in [-1,1]$,
$\zeta = \sinh\mu \in [0,\infty)$,
$\rho = \sqrt{x^2+y^2} = a\sqrt{(1-\xi^2)(1+\zeta^2)}$.

where ρ is the usual cylindrical coordinate. In terms of these

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = a\xi\zeta,$$

A surface of constant μ or ζ is clearly bounded, while those of constant ν are not. In fact $\mu = \mu_0$ is the surface

$$\frac{x^2+y^2}{a^2\cosh^2\mu_0} + \frac{z^2}{a^2\sinh^2\mu_0} = 1,$$

so $A = a\cosh\mu_0$, $B = a\sinh\mu_0$, $B/A = \tanh\mu_0 = \zeta_0/\sqrt{1+\zeta_0^2}$ and $a = \sqrt{A^2-B^2}$ will give our conductor at $\zeta = \zeta_0$.

To find the laplacian, we write the measure

$$\begin{aligned} ds^2 &= dx^2+dy^2+dz^2 = (d\rho)^2 + (dz)^2 \\ &= a^2\Big[\Big(-\frac{\xi}{\sqrt{1-\xi^2}}\sqrt{1+\zeta^2}d\xi + \frac{\zeta}{\sqrt{1+\zeta^2}}\sqrt{1-\xi^2}d\zeta\Big)^2 + \big(\xi\ d\zeta + \zeta\ d\xi\big)^2\Big] \\ &= (h_\xi\ d\xi)^2 + (h_\zeta\ d\zeta)^2 + (h_\phi\ d\phi)^2, \end{aligned}$$

with

$$h_\zeta = a\sqrt{\frac{\zeta^2+\xi^2}{1+\zeta^2}}, \quad h_\xi = a\sqrt{\frac{\zeta^2+\xi^2}{1-\xi^2}}, \quad h_\phi = a\sqrt{(1-\xi^2)((1+\zeta^2)} = \rho.$$

In general the Laplacian is

$$\nabla^2 = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial q_i}\frac{h_1h_2h_3}{h_i^2}\frac{\partial}{\partial q_i},$$

so the equation for the electrostatic potential $\nabla^2 V = 0$ reads $1/a^2(\zeta^2+\xi^2)$ times

$$\frac{\partial}{\partial\xi}\left[\left(1-\xi^2\right)\frac{\partial V}{\partial\xi}\right] + \frac{\partial}{\partial\zeta}\left[\left(1+\zeta^2\right)\frac{\partial V}{\partial\zeta}\right] + \frac{\xi^2+\zeta^2}{(1-\zeta^2)(1+\xi^2)}\frac{\partial^2 V}{\partial\phi^2} = 0. \quad (1)$$

Let us look for a solution by separation of variables,

$$V(\xi,\zeta,\phi) = X(\xi)Z(\zeta)\Phi(\phi),$$

so dividing (1) by V gives

$$\begin{aligned} &\frac{1}{X(\xi)}\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{dX(\xi)}{d\xi}\right] + \frac{1}{Z(\zeta)}\frac{d}{d\zeta}\left[\left(1+\zeta^2\right)\frac{dZ(\zeta)}{d\zeta}\right] \\ &\qquad + \frac{\xi^2+\zeta^2}{(1-\zeta^2)(1+\xi^2)}\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0. \end{aligned}$$

Only the last term can depend on ϕ, so it cannot, and we must have $(\Phi(\phi))^{-1}d^2\Phi(\phi/d\phi^2 = C$, a constant, with exponential solutions. But ϕ is defined only modulo 2π, so we require $\Phi(\phi) = \Phi(\phi+2\pi)$, which requires $C = -m^2, m \in \mathbb{Z}$.

The boundary conditions for our conducting spheroid is that $V = V_0$ at $\zeta = \zeta_0$ for all ξ and ϕ, so in particular we need $\Phi(\phi)$ is a constant, $m = 0$, and then the ζ and ξ dependent terms decouple, we also need $X(\xi)$ to be constant, and our problem reduces to $V = Z(\zeta)$ with

$$(1+\zeta^2)\frac{dZ}{d\zeta} = -K \Longrightarrow Z = -\int d\zeta\frac{K}{1+\zeta^2} = -K\cot^{-1}\zeta + K_2,$$

where K and K_2 are constants. At $\vec r \to \infty$, which is $\zeta \to \infty$, $\cot^{-1}(\zeta) \to 0$, we set the potential to zero, so $K_2 = 0$ with $\cot^{-1}(\zeta)$ defined into $[0,\pi/2]$ for $\zeta \in [0,\infty)$.

From $V = V_0$ at $\zeta = \zeta_0$, we have

$$V = V_0\frac{\cot^{-1}\zeta}{\cot^{-1}\zeta_0}.$$

(b) Consider varying our problem's spheroid, letting $\zeta_0 \to 0$, which reduces our spheroid to a thin disk of radius a. Then $\cot^{-1}\zeta_0 \to \pi/2$,

$$V = \frac{2V_0}{\pi}\cot^{-1}\zeta \underset{r\to\infty}{\longrightarrow} \frac{2V_0}{\pi}\cot^{-1}\frac{r}{a} \approx \frac{2V_0 a}{\pi r},$$

because $r^2 = a^2\left[\left((1+\zeta^2)\left(1-\xi^2\right) + \zeta^2\xi^2\right)\right] \underset{\zeta\to\infty}{\longrightarrow} a^2\zeta^2$. At large distances the potential is $Q/4\pi\epsilon_0 r$, so the total charge on the disk is $Q = 8\epsilon_0 a\,V_0$, and the capacitance is $C = Q/V_0 = 8\epsilon_0 a$.

B Consider the coordinates η and ξ for the plane with

$$x = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}, \qquad y = \frac{a\sin\xi}{\cosh\eta-\cos\xi}.$$

Then

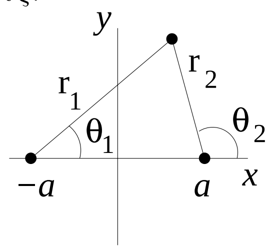
$$(x - a\coth\eta)^2 + y^2 = a^2 \operatorname{csch}^2\eta,$$

so constant η contours are circles centered at $x = a\coth\eta,\ y = 0$, while

$$x^2 + (y - a\cot\xi)^2 = a^2\csc^2\xi,$$

so constant ξ are circles centered at $x = 0,\ y = a\cot\xi$.

The connection of ξ and η to the distances r_1 and r_2 and the angles θ_1 and θ_2 from the points $(\pm a, 0)$, as shown, can be found from



$$\begin{aligned} r_1^2 &= x^2 + 2ax + a^2 + y^2 \\ r_2^2 &= x^2 - 2ax + a^2 + y^2 \end{aligned}$$

$$\begin{aligned} \text{so}\quad r_1^2 + r_2^2 &= 2x^2 + 2a^2 + 2y^2 = 2a^2\frac{\sinh^2\eta + (\cosh\eta - \cos\xi)^2 + \sin^2\xi}{(\cosh\eta - \cos\xi)^2} \\ &= 2a^2\frac{\sinh^2\eta + \cosh^2\eta - 2\cosh\eta\cos\xi + \cos^2\xi + \sin^2\xi}{(\cosh\eta - \cos\xi)^2} \\ &= 4a^2\frac{\cosh\eta}{\cosh\eta - \cos\xi}. \\ r_1^2 - r_2^2 &= 4ax = 4a^2\frac{\sinh\eta}{\cosh\eta - \cos\xi}. \end{aligned}$$

Thus

$$\begin{aligned} r_1^2 &= 2a^2\frac{\cosh\eta + \sinh\eta}{\cosh\eta - \cos\xi} = 2a^2 e^{\eta}\frac{1}{\cosh\eta - \cos\xi} \\ r_2^2 &= 2a^2\frac{\cosh\eta - \sinh\eta}{\cosh\eta - \cos\xi} = 2a^2 e^{-\eta}\frac{1}{\cosh\eta - \cos\xi} \\ \text{so}\quad r_1/r_2 &= e^{\eta}. \end{aligned}$$

From the law of cosines for the triangle shown, $4a^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$ so

$$\begin{aligned} \cos(\theta_2 - \theta_1) &= \frac{(r_1^2 + r_2^2) - 4a^2}{2r_1r_2} \\ &= \left(\frac{\cosh\eta}{\cosh\eta - \cos\xi} - 1\right)\cosh\eta - \cos\xi = \cos\xi, \end{aligned}$$

so $\xi = \theta_2 - \theta_1$, $\eta = \ln(r_1/r_2)$.

Let's get the metric tensor:

$$\begin{aligned} \frac{dx}{a} &= \frac{\cosh\eta\, d\eta}{\cosh\eta - \cos\xi} - \frac{\sinh\eta}{(\cosh\eta - \cos\xi)^2}(\sinh\eta\, d\eta + \sin\xi\, d\xi) \\ &= \frac{(1 - \cosh\eta\cos\xi)d\eta + \sinh\eta\sin\xi\, d\xi}{(\cosh\eta - \cos\xi)^2} \end{aligned}$$

$$\begin{aligned} \frac{dy}{a} &= \frac{(\cos\xi(\cosh\eta - \cos\xi) - \sin^2\xi)d\xi - \sinh\eta\sin\xi\, d\eta}{(\cosh\eta - \cos\xi)^2} \\ &= \frac{(\cos\xi\cosh\eta - 1)d\xi - \sinh\eta\sin\xi\, d\eta}{(\cosh\eta - \cos\xi)^2} \end{aligned}$$

so

$$\begin{aligned} \frac{(dx)^2 + (dy)^2}{a^2} &= \frac{(1 - \cosh\eta\cos\xi)^2 + \sinh^2\eta\sin^2\xi}{(\cosh\eta - \cos\xi)^4}\left((d\eta)^2 + (d\xi)^2\right) \\ &= \frac{(d\eta)^2 + (d\xi)^2}{(\cosh\eta - \cos\xi)^2} \end{aligned}$$

so

$$g_{\eta\eta} = g_{\xi\xi} = a/(\cosh\eta - \cos\xi), \quad g_{\eta\xi} = 0.$$

Consider two circular wires of radius R with their centers separated by a distance $2L$, at $x = \pm L,\ y = 0$. The potential problem in which they carry voltages $\pm V$ satifies the two-dimensional Laplace equation $\nabla^2\psi = 0$ with $\phi = \pm V$ on the circles representing the boundaries of the two wires, at $(x \mp L)^2 + y^2 = R^2$. The positions on the x axis are for $\xi = 0$ or $\xi = \pi$, so

$$L \pm R = \frac{a\sinh\eta}{\cosh\eta \mp 1} \Longrightarrow L^2 - R^2 = a^2, \tag{2}$$

and

$$\frac{L + R}{L - R} = \frac{\cosh\eta + 1}{\cosh\eta - 1} \Longrightarrow \cosh\eta_B = \frac{L}{R}. \tag{3}$$

for η_B the value of $|\eta|$ on the boundaries.

As the Laplacian is

$$g^{\mu\nu}\partial_\mu\partial_\nu = (\cosh\eta - \cos\xi)\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2}\right),$$

Laplace's equation becomes simply

$$\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2}\right)\psi(\eta, \xi) = 0,$$

with boundary conditions

$$\psi(\eta, \pi) = \psi(\eta, -\pi), \quad \psi(\pm\eta_B, \xi) = \pm V/2,$$

with V the maximum value of the voltage between the wires. Thus ψ has the unique solution $\psi = V\eta/2\eta_B$.

To calculate the impediance of this cable, we will calculate the average power flow

$$\begin{aligned} \langle P\rangle &= \hat{z}\cdot\int_A \frac{1}{2}(\vec{E}\times\vec{H}^*) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\int_A \hat{z}\cdot(\vec{E}\times(\hat{z}\times\vec{E}^*)) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\int_A |\vec{E}_t|^2 \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\int_A (\vec{\nabla}_t\psi)^2 = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\int_A \vec{\nabla}_t\cdot\left(\psi\vec{\nabla}_t\psi\right) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\oint_{\partial A}\psi\vec{\nabla}_t\psi \\ &= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}2\,(2\pi)\frac{V}{2}\frac{V}{2\eta_B} = \frac{\pi}{2}\sqrt{\frac{\epsilon}{\mu}}V^2/\cosh^{-1}(L/R) \end{aligned}$$

In the first equality in the second line the $\psi\nabla^2\psi$ term from integration by parts is zero as ψ satisfies Laplace's equation. In the evaluation of the boundary integral in the third equation, the contributions from $\xi = \pm\pi$ cancel.

To find the impediance we write $\langle P\rangle = \langle V^2\rangle/Z = V^2/2Z$, so

$$Z = \sqrt{\frac{\mu}{\epsilon}}\frac{\eta_B}{\pi} = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\cosh^{-1}(L/R) = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(L/R + \sqrt{\frac{L^2}{R^2} - 1}\right).$$

There is another, perhaps more intuitive, way to find the impedance, as $Z = V/I$, where I is the current in one of the wires ($-I$ is in the other). How do we find I? We can use Ampère's law

$$\mu_0 I = \oint \vec{B}\cdot d\vec{\ell} - \frac{1}{c^2}\int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A},$$

where the path is up the y axis from $-\infty$ to ∞ and around a semicircle at ∞. As $E_z = 0$ the displacement current doesn't contribute, and as each wire gives a B field $\propto r^{-1}$, but in opposite directions, the B field falls off like $1/r^2$ at infinity and the semicircle integral goes to zero. Thus

$$\begin{aligned} I &= \pm\frac{1}{\mu_0}\int_{-\infty}^{\infty} dy\, B_y(0,y) = \pm\sqrt{\frac{\epsilon_0}{\mu_0}}\int_{-\infty}^{\infty} dy\, E_x(0,y) = \mp\sqrt{\frac{\epsilon_0}{\mu_0}}\int_{-\infty}^{\infty} dy\,\frac{\partial\psi}{\partial x}(0,y) \\ &= \mp\sqrt{\frac{\epsilon_0}{\mu_0}}\int_{-\infty}^{\infty} dy\left(\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x}\right) = \mp\sqrt{\frac{\epsilon_0}{\mu_0}}\int_{-\infty}^{\infty} dy\left(0 + \frac{V}{2\eta_B}\left.\frac{\partial\eta}{\partial x}\right|_{x=0}\right). \end{aligned}$$

Now

$$\eta = \ln\frac{r_1}{r_2} = \frac{1}{2}\ln\frac{(x+a)^2 + y^2}{(x+a)^2 + y^2}$$

so

$$\frac{\partial\eta}{\partial x} = \frac{1}{2}\left(\frac{2(x+a)}{(x+a)^2 + y^2} - \frac{2(x-a)}{(x-a)^2 + y^2}\right) \xrightarrow[x\to 0]{} \frac{2a}{y^2 + a^2},$$

so, not worrying about overall sign, we have

$$I = \sqrt{\frac{\epsilon_0}{\mu_0}}\frac{V}{\eta_B}\int_{-\infty}^{\infty} dy\,\frac{a}{y^2 + a^2} = \sqrt{\frac{\epsilon_0}{\mu_0}}\frac{\pi}{\eta_B}V.$$

Thus

$$Z = V/I = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\eta_B}{\pi}.$$

If we assume the medium outside the wires is vacuum, and take as the dimensions $R = 0.5$ mm, $L = 4$ mm, so (2) gives $a = 3.9686$ mm, and (3) gives $\eta_B = 2.76866$. Noting that the "impediance of the vacuum"

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.73\ \Omega,$$

we have $Z = 332\ \Omega$.

The dimensions given are about right for old "300 ohm" television cable. That cable, however, does not consist of just two circular wires in vacuum, but includes a plastic coating including a connecting piece which, I guess, has a high dielectric constant. This will lower the impedance.