Intro to Last Lecture (27) (Dec. 14, 2016)

Last time we discussed parallel transport and the covariant derivative. We found on vectors (contravariant) and 1-forms ∼ covariant vectors) we have

$$\begin{aligned}(D_\mu V)^\rho &= \partial_\mu V^\rho + \Gamma^\rho{}_{\nu\mu} V^\nu \\ (D_\mu \mathbf{A})_\nu &= \partial_\mu A_\nu - \Gamma^\rho{}_{\nu\mu} A_\rho .\end{aligned}$$

I should have mentioned that requiring the covariant derivative of a particle's 4-velocity to be zero along its path,

$$\left(\frac{D}{D\tau}u\right)^\nu = u^\rho D_\rho u^\nu = \frac{dq^\rho}{d\tau}\left[\frac{\partial u^\nu}{\partial x^\rho} + \Gamma^\nu{}_{\rho\mu} u^\mu\right] = \frac{d^2 q^\rho}{d\tau^2} + \Gamma^\nu{}_{\rho\mu}\frac{dq^\rho}{d\tau}\frac{dq^\mu}{d\tau} = 0$$

is the geodesic equation.

Today

We will discuss geodesic deviation, which is to say how the vector that describes two neighboring freely-falling particles changes with time. This will lead us to the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$. We will see that the commutator of two covariant derivatives acting on a vector transforms the vector by a matrix which is the curvature tensor. This might remind you of the way the electromagnetic field-strength tensor $F_{\mu\nu} = -iq^{-1}[D_\mu, D_\nu]$, where $D_\mu = \partial_\mu + iqA_\mu$ is the gauge covariant derivative in terms of the electromagnetic vector potential A_μ. It is even closer to the same thing in non-Abelian gauge theories, where A is a field that takes on values which are generators of group transformations in the internal symmetry group, rather than just ordinary vectors[1]. For general relativity, it is the Lorentz transformations of the tangent frames (the ξ^α's of the vierbeins) which are the generators.

Next we will ask what happens to conserved currents, both the electromagnetic J^μ and the energy-momentum tensor $T^{\mu\nu}$. In the absence of gravity these are conserved, so in a local inertial frame they should still be conserved, and more generally as well, providing we use the covariant derivative. For the electromagnetic current we will see that the $\sqrt{g}$ change in the divergence is just what is required to properly integrate over all space to get the total charge. But when we do the same for $T^{\mu\nu}$, we find that setting $D_\mu T^{\mu\nu} = 0$ tells us that dP^j/dt is not just an integral of a total divergence but has an additional term, so the integral of the momentum density given by the energy-momentum tensor of all the matter[2] is not conserved. This is because the gravitational field does change the momentum of the matter.

[1] Actually the i on electromagnetism's A is the generator of a rotation of the complex quantum wave function ψ, that is, a phase transformation.

[2] Including the electromagnetic fields.

As the current density is given by the divergence of the field-strength in the absence of gravity,

$$J^\mu = \partial_\nu F^{\mu\nu} = \partial_\nu \partial^\nu A^\mu - \partial_\nu \partial^\mu A^\nu = A^{\nu,\mu}{}_{,\nu} - A^{\mu,\nu}{}_{,\nu}$$

we might think the equivalence principle should say $J^\mu = D_\nu F^{\mu\nu} = D_\nu(D^\nu A^\mu - D^\mu A^\nu)$, which is correct. But the flat space expression could have been written $J^\mu = \partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = A^{\mu,\nu}{}_{,\nu} - A^\nu{}_{,\nu}{}^{,\mu}$, we might think the equivalence principle would say $J^\mu = D_\nu D^\nu A^\mu - D^\mu D_\nu A^\nu$, but the difference of the two expressions is $D^\mu D_\nu A^\nu - D_\nu D^\mu A^\nu = [D^\mu, D_\nu]A^\nu$, which is not zero. The matrix acting on A is the Ricci tensor $R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}$. Further contraction gives the curvature scalar $R := R_\mu{}^\mu$.

Now it is very disturbing to think that momentum might not be conserved — indeed, doesn't Noether guarantee that translation invariance of the basic laws requires it? So as we want to insist that the total energy-momentum tensor $T^{\mu\nu}$ be covariantly conserved, we will need to add the contribution of the gravitational fields. There are only two tensors which we could add to $T_{\mu\nu}$ which are covariantly conserved, the metric tensor and $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. So if we look for an equation connecting the graviational fields with the matter, it must be

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Thus we will have found the Einstein field equations of general relativity.

If time permits, we should also discuss the wave equation for the gravitational fields, and in particular the gravitons, which are the analogs of photons. Counting the numbers of degrees of freedom is tricky, it turns out there are only two for each wavenumber $\vec{k}$. We might also discuss how to derive the Einstein equation from an action by the Euler-Lagrange method. It turns out that the lagrangian density is very simple:

$$\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G} R - \frac{1}{8\pi G} \Lambda,$$

where G is the Newton's gravitional constant and Λ is the cosmological constant.

- Last class, Wednesday Dec. 14 at noon, here as usual
- homework #11 is voluntary, will not be collected, and the solution has been posted.
- final exam Dec. 20 at 8:00 AM in our usual room, Hill 009. I will try to be more reasonable about the timing, but the exam is three hours. You may bring 3 pages, $11 \times 8\frac{1}{2}$ inches, with *handwritten* notes (on both sides, if you like), but no other materials.