

## Intro to Lecture 26

(Dec. 9, 2016)

Last time we showed that space-time is a curved rather than a flat four dimensional space, and used the equivalence principle to find the gravitational red shift and to argue that light should be deflected by passing near heavy objects. The equivalence principle says that at each point in space-time one can erect a Minkowski coordinate system which is inertial, in the sense that the laws of physics expressed at that point, are what they would be without gravity, but they only hold in an infinitesimal neighborhood of that point. We then introduced the Vierbein fields, the transition function from those inertial charts with coordinates  $\xi^\alpha$  to a more general set of manifold coordinates  $x^\mu$ ,

$$V^\alpha{}_\mu(\mathcal{P}) = \left. \frac{\partial \xi^\alpha}{\partial x^\mu} \right|_{\mathcal{P}}.$$

Then we have the metric tensor

$$\mathbf{g} = \mathbf{d}\xi^\alpha \otimes \mathbf{d}\xi^\beta \eta_{\alpha\beta} = g_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$$

and so we find

$$g_{\mu\nu} = \eta_{\alpha\beta} V^\alpha{}_\mu V^\beta{}_\nu \quad \text{and} \quad \Gamma^\rho{}_{\mu\nu} := (V^{-1})^\rho{}_\alpha V^\alpha{}_{\mu,\nu} = \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

for the affine connection.

Comparing the geodesic equation with that of a particle falling in Newtonian gravity we found that for nonrelativistic motion  $g_{00} = 1 - 2\phi$ , where  $\phi$  is the gravitational potential (or  $2\phi = c^2(1 - g_{00})$  if you can't accept  $c = 1$ ).

Today

We turn to the question of parallel transport, or really the question of whether two vectors at nearby points are the same or not. We begin by observing that in the inertial chart at  $\mathcal{P}$ , parallel transport of a vector at  $\mathcal{P}$  through an infinitesimal distance leaves its components unchanged. Describing this infinitesimal parallel transport in some other chart defines the covariant derivative. Covariantly differentiating a scalar function is just ordinary differentiation, and as a 1-form acting on a vector is a scalar, and the derivatives all obey the product rule, we can find the rules for covariantly differentiating 1-forms as well, and then an arbitrary tensor.

We will see how these affect the differential objects, leaving the gradient and curl unchanged but affecting the divergence. Finally, in preparation for exploring curvature next class, but also to show these ideas have wider application, we will describe the analogous covariant derivative in electromagnetism, which involves the vector potential  $A^\mu$ , and perhaps even consider non-Abelian gauge theory, where the  $A^\mu(x)$  is not a real number but a generator of a group transformation. In both cases, considering the integral of the covariant derivative around an infinitesimal loop gives a variation by the field

strength 2-form  $\mathbf{F}$  or tensor  $F^{\mu\nu}$ . Following the analogue back to space-time, where the  $A$  field represents Lorentz transformations of the inertial frame  $\xi^\alpha$ , we will see how the curvature  $R^\mu{}_{\nu\rho\sigma}$  is analogous to  $F^{\mu\nu}$ .

In the last lecture, we will discuss geodesic deviation, which is to say how the vector that describes two neighboring freely-falling particles changes with time. This will lead us to the Riemann curvature tensor  $R^\rho{}_{\sigma\mu\nu}$ . By insisting that the energy-momentum tensor  $T^{\mu\nu}$  be covariantly conserved, we find the field equations that tells space-time how to curve in the presence of matter. This involves pieces of  $R^\rho{}_{\sigma\mu\nu}$ . Thus we will have found the Einstein field equations of general relativity.

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- Last class, Wednesday Dec. 14 at noon, here as usual
- homework #11 is voluntary, will not be collected, but the solution will be posted Tuesday.
- final exam Dec. 20 at 8:00 AM in our usual room, Hill 009. I will try to be more reasonable about the timing, but the exam is three hours. You may bring 3 pages,  $11 \times 8\frac{1}{2}$  inches, with *handwritten* notes (on both sides, if you like), but no other materials.