Intro to Lecture 17 Nov. 4, 2016

Last time we considered the general formulation for solutions of a second order linear differential equation for $y(x)$

$$\frac{d}{dx}p(x)\frac{dy(x)}{dx} + q(x)y(x) + \lambda w(x)y(x) = 0,$$

with regular singular points a and b and with $p(x)$, $q(x)$ and $w(x)$ analytic on the interval (a, b), and $w(x) > 0$ on (a, b). Because the undetermined separation constant can take on many values (generally a discrete but infinite set of them), we may consider the vector space of all the homogeneous solutions, and define an inner product, given by the integral with weight x,

$$\langle v, u\rangle := \int w(x)\, v^*(x)u(x)\, dx.$$

Then we may consider the second order differential equation as an eigenvalue problem for the second order differential operator acting on the space of solutions. Then the solutions form an orthonormal basis of an infinite dimensional vector space, and we can use the techniques and knowledge we have from linear algebra, albeit with some caution as we are now in an infinite-dimensional vector space.

Today we will apply these ideas to discuss orthogonality of the solutions with the weighted measure, and *Bessel's inequality* describing how closely an expansion of an arbitrary function $F(x)$ in terms of these eigenfunctions matches the function, and when this expansion is *complete*. We will also see how the Dirac delta function $\delta(x_1 - x_2)$ and the Green's function $G(x_1, x_2)$ are given by products $\phi_n(x_1)\phi_n(x_2)$ of eigenfunctions. The Green's function allows us to solve the inhomogeneous equation, when we have a given source term, as in Poisson's equation.

We will then turn to finding the *classical orthogonal polynomials* corresponding to the weight functions w of our most important equations. These polynomials will give solutions to the equations with $q = 0$. Each set of polynomials, corresponding to each $p(x)$ and $w(x)$ which we encounter, has a generating function and a recursion relation which enable simple calculations of the polynomials.

- Homework #7 is due Monday at 5:00 PM as usual.
