0.1 Wronskian

A set of n vectors $\vec{V}_j$ is linearly independent if the only set $a_j \in F$ such that $\sum_j a_j \vec{V}_j = 0$ is $a_j = 0$ for all j. If the vectors are in an n dimensional space and they are given in terms of a set of basis vectors, we have $\sum_j a_j \left(V_j\right)_k = 0$ for all k, which means the matrix $M_{kj} = \left(V_j\right)_k$ annihilates the vector a_j, and the $\vec{V}_j$ are linearly independent if and only if $\det M = 0$.

If we have a set of n linearly independent solutions $y_k(x)$ to an n'th order ordinary linear differential equation, the Wronskian is defined as the determinant of the matrix of j'th derivatives, $j = 0, \ldots, n-1$ of the n functions y,

$$W(x) := \det \frac{d^{\ell-1} y_k}{dx^{\ell-1}} = \sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \prod_{j=0}^{n-1} \frac{d^j y_{i_j}}{dx^j},$$

where by $d^0 y/dx^0$ we just mean y. If we had a set $a_k(x)$ such that $\displaystyle\sum_k a_k(x) \frac{d^{\ell-1} y_k}{dx^{\ell-1}}(x) = 0$ for all ℓ, without the $a_k(x)$ all vanishing, we would have a linear dependence among our n solutions, because the n'th derivative would also vanish, as each y_k satisfies the equation. So linear independence tells us the determinant does not vanish.

If we differentiate W, we have

$$\frac{dW}{dx} = \sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \sum_{\ell=0}^{n-1} \prod_{j=0}^{n-1} \frac{d^{j+\delta_{j\ell}} y_{i_j}}{dx^{j+\delta_{j\ell}}}.$$

Note that unless $\ell = n-1$, the terms in the product with $j = \ell$ and the terms with $j = \ell + 1$ are now identical, except for interchanging the indices on y, so the ϵ kills them, and we have only the contribution from $\ell = n-1$, which is

$$\frac{dW}{dx} = \sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \frac{d^n y_{i_n}}{dx^n} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j}.$$

Now if our differential equation is

$$\frac{d^n y}{dx^n}(x) + P(x) \frac{d^{n-1} y}{dx^{n-1}}(x) + \sum_{\ell=0}^{n-2} Q_\ell(x) \frac{d^\ell y}{dx^\ell}(x) = 0$$

we can substitute the values of $d^n y_{i_n}/dx^n$,

$$\begin{aligned}
\frac{dW}{dx} &= -\sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j} \\
&\qquad \times \left(P(x) \frac{d^{n-1} y_{i_n}}{dx^{n-1}}(x) + \sum_{\ell=1}^{n-2} Q_\ell(x) \frac{d^\ell y_{i_n}}{dx^\ell}(x) \right) \\
&= -\sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \prod_{j=0}^{n-1} \frac{d^j y_{i_j}}{dx^j} P(x) - \sum_{i_1,\ldots,i_n} \sum_{\ell=1}^{n-2} Q_\ell(x) \epsilon_{i_1,\ldots,i_n} \frac{d^\ell y_{i_n}}{dx^\ell} \prod_{j=0}^{n-2} \frac{d^j y_{i_j}}{dx^j}
\end{aligned}$$

Note that in the terms involving Q_ℓ, the term before the product matches one of the terms in the product, except for interchange $i_n \leftrightarrow i_\ell$, and so they are killed by the ϵ. Also notice the term multiplying $-P(x)$ is just $W(x)$, so we have the first order differential equation

$$\frac{dW}{dx} = -P(x) W(x).$$