Rodrigues' Formula and Orthogonal Polynomials

Suppose we have a weight function $w > 0$ on (a,b), with $\int_a^b w(x)x^n\,dx$ defined for all $n \in \mathbb{N}$. Then we can define a sequence of orthogonal polynomials $f_n(x)$ of order n such that

$$\int_a^b w(x) f_n(x) f_m(x)\,dx = h_n \delta_{mn}.$$

This can be done iteratively by a kind of Schmidt diagonalization.
[Doesn't this mean that for given $h_n > 0$ the f_n's are determined (up to sign)? And then they don't depend on $g(x)$?]

We want the $f_n(x)$ to satisfy the Sturm-Liouville equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \lambda_n w(x) y(x) = 0. \tag{1}$$

where we expect to have $p(x) = w(x)g(x)$ and $p(a) = p(b) = 0$, $p(x) > 0$ on (a,b). Notice that $p(x)$ and $w(x)$ are always closely related.

Actually, if we have n'th order polynomial solutions $f_n(x)$ for all $n \in \mathbb{N}$, we automatically have $\lambda_0 = 0$ and

$$\frac{d}{dx} g(x)w(x) = (-\lambda_1 x + k) w(x), \tag{2}$$

so indeed $p(x)$ is determined up to an additive constant by $w(x)$ and λ_1 and k.

We claim that, under suitable conditions on $w(x)$ and $g(x)$, the *Rodrigues' Formula*

$$f_n(x) = \frac{1}{a_n w(x)} \frac{d^n}{dx^n}\left(w g^n\right). \tag{3}$$

where a_n is a non-zero constant, gives us the f_n which satisfy both conditions.

Actually, we might write this as

$$f_n(x) = \frac{g}{a_n} D^n g^{n-1} \quad \text{with} \quad D := \left[\frac{1}{gw}\frac{d}{dx} g w\right] = \left[\frac{1}{g}(-\lambda_1 x + k) + \frac{d}{dx}\right] \tag{4}$$

The conditions we expect to need are

$$\frac{d^{r-1}}{dx^{r-1}} w(x) g^n(x) \underset{\substack{x\to a \\ \text{or } x\to b}}{\longrightarrow} 0, \qquad \text{for} \quad r \le n \tag{5}$$

for all $n \in \mathbb{Z}^+$. First, $f_n(x)$ is a polynomial of order n. Proof: $f_0 = 1/a_0$, and as $f_n(x) = \dfrac{g}{a_n}\left(\dfrac{1}{gw}\dfrac{d}{dx} g w(x)\right)^n g^n = \dfrac{g}{a_n}\left[\dfrac{1}{g}(-\lambda_1 x + k) + \dfrac{d}{dx}\right]^n g^n$. Let $D := \left[\dfrac{1}{g}(-\lambda_1 x + k) + \dfrac{d}{dx}\right]$ so $f_n(x) = \dfrac{g}{a_n} D^n g^{n-1}$.

If we assume $g(x)$ is a non-zero polynomial of order at most 2, and if $\phi_0(x)$ is a polynomial of order $\le q$,

$$D g^r \phi_0 = (-\lambda_1 x + k + r\frac{dg}{dx}) g^{r-1}\phi_0 + g^r \frac{d\phi_0}{dx}$$

which is $g^{r-1}\phi_1(x)$ where $\phi_1(x) = (-\lambda_1 x + k + r\frac{dg}{dx})\phi_0 + g\frac{d\phi_0}{dx}$ is a polynomial of order $\le q+1$, so applying this $n-1$ times to g^n gives a polynomial of order $\le n-1$, and then applying gD to this is a polynomial of order n.

We have already assumed (5): $\dfrac{d^m}{dx^m}(wg^n) \underset{a \text{ or } b}{\longrightarrow} 0$ for $m < n$ which we need to show that f_n is orthogonal to any polynomial $p(x)$ of order $< n$, as

$$\begin{aligned}
\langle f_n, p\rangle &= \int_a^b w(x) f_n(x) p(x)\,dx = \frac{1}{a_n}\int_a^b p(x)\frac{d^n}{dx^n}\left(wg^n\right)\,dx \\
&= \underbrace{\frac{1}{a_n}\, p(x)\frac{d^{n-1}}{dx^{n-1}}\left(wg^n\right)\Big|_a^b}_{=0} - \frac{1}{a_n}\int \frac{dp}{dx}\frac{d^{n-1}}{dx^{n-1}}\left(wg^n\right)\,dx \\
&= \cdots = (-1)^n \frac{1}{a_n}\int \frac{d^n p}{dx^n}\,\left(wg^n\right)\,dx = 0
\end{aligned}$$

as $p(x)$ is of order $< n$. The boundary terms vanished by the condition (5). So in particular $\langle f_n, f_m\rangle = h_n \delta_{nm}$ for some positive h_n.

To see that f given by Eq. (3) satisfies Eq.(1), note that

$$\begin{aligned}
\left(\frac{d}{dx}\right)^{n+1} g\frac{d}{dx} w g^n &= g\left(\frac{d}{dx}\right)^{n+2} w g^n + (n+1)\frac{dg}{dx}\left(\frac{d}{dx}\right)^{n+1} w g^n \\
&\quad + \frac{n(n+1)}{2}\frac{d^2 g}{dx^2}\left(\frac{d}{dx}\right)^n w\, g^n
\end{aligned} \tag{6}$$

as g has no higher derivatives (g is quadratic).

Note $g\frac{d}{dx}wg^n = \frac{d(wg)}{dx}g^n + (n-1)wg^n\frac{dg}{dx} = wg^n\left(-\frac{\lambda_1 f_1}{k_1} + (n-1)\frac{dg}{dx}\right)$
The term in parenthesis is linear, so at most one of the $n+1$ derivatives acts on it, and

$$\begin{aligned}\left(\frac{d}{dx}\right)^{n+1} g\frac{d}{dx}wg^n &= \left[-\frac{\lambda_1}{k_1}f_1 + (n-1)\frac{dg}{dx}\right]\left(\frac{d}{dx}\right)^{n+1} wg^n \\ &\quad +(n+1)\left(-\lambda_1 + (n-1)\frac{d^2g}{dx^2}\right)\left(\frac{d}{dx}\right)^n wg^n.\end{aligned} \tag{7}$$

Equating the right hand sides of Eqs. (6) and (7) and using Eq. (2)

$$\left\{g\left(\frac{d}{dx}\right)^2 + \left(2\frac{dg}{dx} + \frac{\lambda_1}{k_1}f_1\right)\frac{d}{dx} + (n{+}1)\left(\frac{2{-}n}{2}\frac{d^2g}{dx^2} + \lambda_1\right)\right\}a_n w f_n = 0,$$

or

$$\left\{gw\frac{d^2}{dx^2} + \underbrace{\left(2g\frac{dw}{dx} + 2w\frac{dg}{dx} + \frac{\lambda_1}{k_1}wf_1\right)}_{\frac{d}{dx}gw}\frac{d}{dx} + \lambda_n w\right\}f_n = 0$$

with

$$\lambda_n = g\frac{1}{w}\frac{d^2w}{dx^2} + \left(2\frac{dg}{dx} + \frac{\lambda_1}{k_1}f_1\right)\frac{1}{w}\frac{dw}{dx} + (n{+}1)\left[\frac{2{-}n}{2}\frac{d^2g}{dx^2} + \lambda_1\right].$$

$$\begin{aligned}\text{From}\quad -\frac{\lambda_1}{k_1}f_1 &= \frac{1}{w}\frac{d}{dx}gw = g' + g\frac{1}{w}\frac{dw}{dx} \\ -\lambda_1 &= g'' + g\frac{1}{w}\frac{d^2w}{dx^2}\underbrace{-g\left(\frac{1}{w}\frac{dw}{dx}\right)^2 + g'\frac{1}{w}\frac{dw}{dx}}_{2g'\frac{1}{w}\frac{dw}{dx} - \frac{\lambda_1}{k_1}f_1\frac{\lambda_1}{k_1}}\end{aligned}$$

so $\lambda_n = -\lambda_1 - g'' + (n+1)\lambda_1 - (n+1)\left(\frac{n}{2} - 1\right)g'' = n\lambda_1 - \frac{1}{2}n(n-1)g''$ and

$$\left(\frac{d}{dx}p\frac{d}{dx} + \lambda_n w\right)f_n = 0.$$