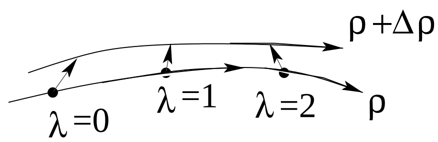
Physics 464/511 Lecture R Fall, 2016

1 Geodesic Deviation, and the Field Equations

Consider a set of trajectories through spacetime, the set parameterized by ρ, the trajectories separated by a small distance. Each trajectory is a curve parameterized by a parameter λ, At each fixed λ, we may consider



$\mathbf{n} = \Delta\mathcal{P}|_\lambda$ as a vector $\Delta\rho\dfrac{\partial\mathcal{P}}{\partial\rho} = \Delta\rho\dfrac{\partial x^\mu}{\partial\rho}\dfrac{\partial\mathcal{P}}{\partial x^\mu} = \Delta\rho\dfrac{\partial x^\mu}{\partial\rho}\partial_\mu$, where x^μ are the coordinates in some chart. We define $n^\mu = \Delta\rho\dfrac{\partial x^\mu}{\partial\rho}$.

Let us ask how $\mathbf{n}$ develops as we move along the trajectories, assuming each of the trajectories obeys the law of geodesic transport:

$$\frac{Du^\nu}{D\lambda} = \left.\frac{\partial u^\nu}{\partial\lambda}\right|_\rho + \Gamma^\nu{}_{\rho\sigma}u^\rho u^\sigma = 0 \qquad \text{with} \quad u^\nu = \left.\frac{\partial x^\nu}{\partial\lambda}\right|_\rho .$$

Now $\left.\dfrac{\partial n^\mu}{\partial\lambda}\right|_\rho = \Delta\rho\dfrac{\partial^2 x^\mu}{\partial\lambda\,\partial\rho} = \Delta\rho\left.\dfrac{\partial u^\mu}{\partial\rho}\right|_\lambda$.

Taking $\Delta\rho\dfrac{\partial}{\partial\rho}$ of the geodesic equation, using $\Delta\rho\dfrac{\partial}{\partial\rho}\dfrac{\partial u^\nu}{\partial\lambda} = \Delta\rho\dfrac{\partial^3 x^\nu}{\partial\rho\partial\lambda^2} = \dfrac{\partial^2}{\partial\lambda^2}n^\mu$, we have

$$\frac{\partial^2 n^\mu}{\partial\lambda^2} + \underbrace{\left.\Delta\rho\,\frac{\partial}{\partial\rho}\Gamma^\mu{}_{\rho\sigma}\right|_\lambda}_{\Delta\rho\frac{\partial x^\alpha}{\partial\rho}\Gamma^\mu{}_{\rho\sigma,\alpha}=n^\alpha\Gamma^\mu{}_{\rho\sigma,\alpha}} u^\rho u^\sigma + 2\Gamma^\mu{}_{\rho\sigma}u^\rho\frac{\partial n^\sigma}{\partial\lambda} = 0. \qquad (1)$$

Let us use this to evaluate the second covariant derivative along the free-

falling path of $\mathbf{n}$, with[1] $\dfrac{Dn^\mu}{D\lambda} = \dfrac{\partial x^\sigma}{\partial\lambda}D_\sigma n^\mu = \dfrac{\partial n^\mu}{\partial\lambda} + u^\sigma\Gamma^\mu{}_{\rho\sigma}n^\rho$,

$$\begin{aligned}
\frac{D}{D\lambda}\frac{Dn^\mu}{D\lambda} &= \frac{D}{D\lambda}\left(\frac{\partial n^\mu}{\partial\lambda} + \Gamma^\mu{}_{\rho\sigma}n^\rho u^\sigma\right)\\
&= \frac{\partial}{\partial\lambda}\left(\frac{\partial n^\mu}{\partial\lambda} + \Gamma^\mu{}_{\rho\sigma}n^\rho u^\sigma\right) + \Gamma^\mu{}_{\alpha\beta}u^\beta\left(\frac{\partial n^\alpha}{\partial\lambda} + \Gamma^\alpha{}_{\rho\sigma}n^\rho u^\sigma\right)\\
\frac{D^2 n^\mu}{D\lambda^2} &= \frac{\partial^2 n^\mu}{\partial\lambda^2} + \Gamma^\mu{}_{\rho\sigma,\beta}n^\rho u^\sigma u^\beta + 2\Gamma^\mu{}_{\rho\sigma}\frac{\partial n^\rho}{\partial\lambda}u^\sigma + \Gamma^\mu{}_{\rho\sigma}n^\rho \quad \underbrace{\frac{\partial u^\sigma}{\partial\lambda}}_{-\Gamma^\sigma{}_{\beta\alpha}u^\beta u^\alpha}\\
&\qquad + \Gamma^\mu{}_{\alpha\beta}\Gamma^\alpha{}_{\rho\sigma}n^\rho u^\beta u^\sigma
\end{aligned}$$

But from Eq. (1)

$$\begin{aligned}
0 &= \frac{\partial^2 n^\mu}{\partial\lambda^2} + \Gamma^\mu{}_{\sigma\beta,\rho}n^\rho u^\sigma u^\beta + 2\Gamma^\mu{}_{\rho\sigma}\frac{\partial n^\rho}{\partial\lambda}u^\sigma\\
\text{so}\quad \frac{D^2 n^\mu}{D\lambda^2} &= \left[\Gamma^\mu{}_{\rho\sigma,\beta} - \Gamma^\mu{}_{\sigma\beta,\rho} + \Gamma^\mu{}_{\beta\alpha}\Gamma^\alpha{}_{\rho\sigma} - \Gamma^\mu{}_{\rho\alpha}\Gamma^\alpha{}_{\beta\sigma}\right]n^\rho u^\beta u^\sigma\\
&= R^\mu{}_{\sigma\rho\beta}n^\rho u^\beta u^\sigma\ ,
\end{aligned}$$

where we have defined[2] the Riemann curvature tensor

$$R^\mu{}_{\sigma\rho\beta} := \Gamma^\mu{}_{\rho\sigma,\beta} - \Gamma^\mu{}_{\sigma\beta,\rho} + \Gamma^\mu{}_{\beta\alpha}\Gamma^\alpha{}_{\rho\sigma} - \Gamma^\mu{}_{\rho\alpha}\Gamma^\alpha{}_{\beta\sigma}.$$

The equation $\dfrac{D^2 n^\mu}{D\lambda^2} = R^\mu{}_{\sigma\rho\beta}n^\rho u^\beta u^\sigma$ is called the equation of *geodesic deviation.*

There is another way to understand the curvature tensor, in terms of the commutator of covariant derivatives. Consider two covariant derivatives acting on a vector V^ρ. First applying D_ν,

$$(D_\nu V)^\rho = V^\rho{}_{,\nu} + \Gamma^\rho{}_{\sigma\nu}V^\sigma$$

and then applying D_μ:

$$\begin{aligned}
(D_\mu D_\nu V)^\rho &= V^\rho{}_{,\nu,\mu} + \Gamma^\rho{}_{\sigma\nu,\mu}V^\sigma + \Gamma^\rho{}_{\sigma\nu}V^\sigma{}_{,\mu}\\
&\quad + \Gamma^\rho{}_{\tau\mu}\left(V^\tau{}_{,\nu} + \Gamma^\tau{}_{\sigma\nu}V^\sigma\right) - \Gamma^\kappa{}_{\nu\mu}\left(V^\rho{}_{,\kappa} + \Gamma^\rho{}_{\sigma\kappa}V^\sigma\right)
\end{aligned}$$

[1]Note: In the next equation, α and β are ordinary space-time indicies, not vierbein tangent space indices. The replacement of $\partial u^\sigma/\partial\lambda$ with $-\Gamma^\sigma{}_{\beta\alpha}u^\beta u^\alpha$ is due to $Du^\nu/D\lambda = 0$.

[2]There seems to be an overall sign discrepancy in $R^\mu{}_{\sigma\rho\beta}$ between authors. Take the following with this in mind.

Subtracting the interchange $\mu \leftrightarrow \nu$, using that $\Gamma^\kappa{}_{\nu\mu}$ is symmetric, we have for the commutator

$$([D_\mu, D_\nu]\, V)^\rho = \left(\Gamma^\rho{}_{\sigma\nu,\mu} - \Gamma^\rho{}_{\sigma\mu,\nu} + \Gamma^\rho{}_{\tau\mu}\Gamma^\tau{}_{\sigma\nu} - \Gamma^\rho{}_{\tau\nu}\Gamma^\tau{}_{\sigma\mu}\right) V^\sigma = R^\rho{}_{\sigma\nu\mu} V^\sigma\ .$$

Notice that this tells us that $R^\rho{}_{\sigma\nu\mu}$ is antisymmetric in its last two indices.

This relation giving the commutator acting on a vector has an interpretation in terms of parallel transporting a vector around a rectangle of dimensions $\Delta x^\mu \times \Delta x^\nu$ in the $\mu\nu$ plane. The vector will be transformed by the matrix $R^\rho{}_{\sigma\nu\mu}$. As the length of the vector will not be changed (as $\mathbf{g}$ is covariantly constant), this matrix will have to be a Lorentz transformation.

This gives an interesting piece of information about a bunch of particles initially at rest in a freely falling inertial chart. Then $\frac{D}{D\lambda} = \frac{d}{dt}$ if we take $\lambda = \tau$, the spatial components give the rule for the acceleration of a particle at separation $\vec{n}$ at rest

$$\left(\frac{d^2\vec{n}}{dt^2}\right)^i = -R^i{}_{0j0} n^j.$$

More on the Equivalence Principle

Let $T^{\mu\nu}$ be the stress-energy tensor of matter (that is, no gravitational contribution to energy density, *etc.*), given by special relativity as a function of the fields, for example, of photons, charged particles, as discussed in Lecture O. We know if we include all such matter and if there is no gravity, $\partial_\mu T^{\mu\nu} = 0$. This must still be true in the local inertial frame even if there is gravity. To make it a statement independent of chart, note that in the inertial frame $D_\mu = \partial_\mu$, so $D_\mu T^{\mu\nu} = 0$. Similarly for the electromagnetic current

$$D_\mu J^\mu = 0 = \partial_\mu J^\mu + \Gamma^\mu{}_{\nu\mu} J^\nu = g^{-1/2}\partial_\mu\left(g^{1/2} J^\mu\right).$$

This changed form for the divergence raises the question of whether charge is conserved! In special relativity we write $Q = \int J^0 d^3V$ and use $\partial_0 J^0 = \vec{\nabla}\cdot\vec{J}$ and $\int \vec{\nabla}\cdot\vec{J} d^3V = \int_S \to 0$ to show $dQ/dt = 0$. We can write this expression in an inertial chart

$$Q = \int_V J^0 dx^1 \wedge dx^2 \wedge dx^3 = \int_V \varepsilon(\mathbf{J},,,)$$

The last expression is entirely geometrical. In an arbitrary frame it reduces to

$$Q = \frac{1}{3!}\int_V \varepsilon_{\mu\nu\rho\sigma} J^\mu dx^\nu \wedge dx^\rho \wedge dx^\sigma = \frac{1}{3!}\int_V \sqrt{g}\,\epsilon_{\mu\nu\rho\sigma} J^\mu dx^\nu \wedge dx^\rho \wedge dx^\sigma.$$

Let us choose V (which can be an arbitrary spacelike hypersurface) to be $t =$ constant, so

$$\begin{aligned} Q &= \int \sqrt{g}\, J^0\, dx^1 \wedge dx^2 \wedge dx^3. \\ \frac{dQ}{dt} &= \int_V \partial_0\left(\sqrt{g}J^0\right) d^3V = -\int \vec{\nabla}\cdot\left(\sqrt{g}\vec{J}\right) d^3V \to 0, \end{aligned}$$

so Q is indeed conserved.

What about energy and momentum?

$$D_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\rho\mu} T^{\rho\nu} + \Gamma^\nu{}_{\rho\mu} T^{\mu\rho} = g^{-1/2}\partial_\mu\left(T^{\mu\nu}\sqrt{g}\right) + \Gamma^\nu{}_{\rho\mu} T^{\mu\rho}.$$

The first term in the last expression is just what's needed to make

$$P^\nu = \int \sqrt{g}\, T^{0\nu}\, d^3V$$

conserved, but the second term breaks the conservation. This is because the gravitational force changes the momentum of the matter.

We have already discussed the form of Maxwell's laws in a geometrical form $\mathbf{d}^*\mathbf{F} = {}^*\mathbf{J}$, which can be written

$$\mathbf{d}\left(\sqrt{g}F^{\mu\nu}\right) \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \epsilon_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta}\sqrt{g}\, J^\mu\, \mathbf{d}x^\nu \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$$

or $\left(\sqrt{g}F^{\mu\nu}\right)_{,\nu} = \sqrt{g}J^\mu$.

But
$$\begin{aligned} D_\nu F^{\mu\nu} &= \partial_\nu F^{\mu\nu} + \underbrace{\Gamma^\mu{}_{\rho\nu} F^{\rho\nu}}_{0} + \Gamma^\nu{}_{\rho\nu} F^{\mu\rho} \\ &= g^{-1/2}\partial_\nu\left(\sqrt{g}F^{\mu\nu}\right) = J^\mu. \end{aligned}$$

Could we have started with an equation for A in special relativity and used the equivalence principle? We start with

$$J^\mu = F^{\mu\nu}{}_{,\nu} = -A^{\mu,\nu}{}_{,\nu} + A^{\nu,\mu}{}_{,\nu}.$$

I can write this covariantly as

$$J^\mu = -A^{\mu;\nu}{}_{;\nu} + A^{\nu;\mu}{}_{;\nu} = -D_\nu D^\nu A^\mu + D_\nu D^\mu A^\nu.$$

But alternatively I could have writen it in flat space as $J^\mu = -A^{\mu,\nu}{}_{,\nu} + A^\nu{}_{,\nu}{}^{,\mu}$ and tried to make it covariant by replacing it by $J^\mu \stackrel{?}{=} -D_\nu D^\nu A^\mu + D^\mu D_\nu A^\nu$. Are they both correct? The difference is $0 \stackrel{?}{=} [D_\nu, D^\mu] A^\nu = R^\nu{}_{\rho\nu}{}^\mu A^\rho$. We define, in general, the

$$\text{Ricci tensor:} \qquad R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}$$

so we see that they are equivalent only if the Ricci tensor vanishes. The correct rule is, of course, the first, which is $D_\nu F^{\mu\nu} = J^\mu$. The second can be ruled out because it is not covariant under *electromagnetic* gauge transformations. This is a warning that using the equivalence principle to replace derivatives by covariant derivatives is only straightforward for the first one, at the point where Γ vanishes. That is, the equivalence principle shouldn't be used with too much blind faith, as it never answers 2nd degree derivative questions. We should not be surprised at its failure here, because we have had to use second derivatives, which will not be the same even in the local inertial frame as they are in flat space.

We also define the

$$\text{Scalar curvature:} \qquad R := R_\mu{}^\mu = g^{\mu\nu} R_{\mu\nu}$$

and the

$$\text{Einstein curvature tensor:} \qquad G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

Bianchi identities involving Ricci and Scalar curvatures:

In homework 2 we found that the operation of commuting elements of an algebra gives a Lie algebra, and the commutation satisfies the Jacobi identity. For the covariant derivatives, this means $D_\rho[D_\mu, D_\nu] + D_\mu[D_\nu, D_\rho] + D_\nu[D_\rho, D_\mu] = 0$ which when applied to an arbitrary vector V^β gives us our first *Bianchi*

Identity:

$$\begin{aligned}
B: &\quad R^\alpha{}_{\beta\mu\nu;\rho} + R^\alpha{}_{\beta\nu\rho;\mu} + R^\alpha{}_{\beta\rho\mu;\nu} = 0 \\
\delta^\mu_\alpha B: &\quad R_{\beta\nu;\rho} + R^\alpha{}_{\beta\nu\rho;\alpha} - R_{\beta\rho;\nu} = 0 \\
\frac{1}{3}\epsilon_{\kappa\lambda\alpha}{}^\beta B \epsilon^{\mu\nu\rho\lambda}: &\quad \epsilon_{\kappa\lambda\alpha}{}^\beta R^\alpha{}_{\beta\mu\nu;\rho}\epsilon^{\mu\nu\rho\lambda} = 0 \\
&\quad = -\delta^{\mu\nu\rho}_{\kappa\alpha\beta} R^{\alpha\beta}{}_{\mu\nu;\rho} = -R^{\alpha\beta}{}_{\kappa\alpha;\beta} - R^{\alpha\beta}{}_{\alpha\beta;\kappa} - R^{\alpha\beta}{}_{\beta\kappa;\alpha} \\
&\quad = 2R^\beta{}_{\kappa;\beta} - R_{;\kappa} = 0
\end{aligned}$$

The curvature tensor has lots of symmetries in its indices, made clearer if we lower the first. Then $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$, that is, symmetric under exchanging the first two with the last two. We have already seen it is antisymmetric in the last two, so must be antisymmetric in the first two as well. Also a cyclicity: $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$, which can also be written $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0$. These are demonstrated in Weinberg, p. 141, starting with the expression in terms of derivatives, first and second, of g.

Define $\mathcal{G}$ to be the double dual of R, that is,

$$\mathcal{G}^{\alpha\beta}{}_{\mu\nu} = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta} R_{\gamma\delta}{}^{\rho\sigma}\epsilon_{\rho\sigma\mu\nu}.$$

Define "Einstein"

$$\begin{aligned}
G^\beta{}_\nu &:= \mathcal{G}^{\alpha\beta}{}_{\alpha\nu} = -\frac{1}{4}\delta^{\beta\gamma\delta}_{\rho\sigma\nu} R_{\gamma\delta}{}^{\rho\sigma} = -\frac{1}{2} R_{\gamma\nu}{}^{\beta\nu} - \frac{1}{2} R_{\gamma\rho}{}^{\rho\beta} = R_{\rho\sigma}{}^{\rho\sigma} \\
&= R_\nu{}^\beta - \frac{1}{2} R\delta^\beta_\nu,
\end{aligned}$$

so $G_{\mu\nu} = R_{\mu\nu} - \dfrac{1}{2} g_{\mu\nu} R$, and the last Bianchi identity is $G^\mu{}_{\nu;\mu} = 0$.

Note $G^\mu{}_\mu = R^\mu{}_\mu - \frac{1}{2}\delta^\mu_\mu R = -R$.

2 Equations Determining Geometry

Mass is the source of gravity in Newtonian mechanics. Matter must affect the metric in some way in general relativity. Let us return to the Sun.

$$\nabla^2\phi = 4\pi G\rho, \quad g_{00} = -1 - 2\phi, \quad g_{\mu\nu,0} = 0, \qquad \text{all } \Gamma \propto \phi,$$

so to first order in ϕ, $R^\alpha{}_{\beta\mu\nu} = 2\Gamma^\alpha{}_{\beta[\nu,\mu]}$, and $R^\alpha{}_{0\mu 0} = \Gamma^\alpha{}_{00,\mu}$. Thus $R_{00} = \Gamma^i{}_{00,i} - \underbrace{\Gamma^0{}_{00,0}}_{0} = -\frac{1}{2}\nabla^2 g_{00} = \nabla^2\phi = 4\pi G\rho$.

ρ is the mass density, or energy density, or T_{00}, which suggests a connection between $R_{\mu\nu}$ and $T_{\mu\nu}$. We have seen that $D_\mu T^{\mu\nu} = 0$ is an equation of motion, at least for particles in an electromagnetic field. So any connection with $T_{\mu\nu} \propto R_{\mu\nu}$ cannot be right, because $D_\mu R^{\mu\nu} \neq 0$. But $D^\mu G_{\mu\nu} \equiv 0$, so perhaps $T_{\mu\nu} \propto G_{\mu\nu}$. But for a point particle at rest, $T_{\mu\nu} = 0$ unless $\mu = \nu = 0$. So $R = -G^\mu{}_\mu = -G^0{}_0$, so $G_{00} = R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} G_{00}$, or $G_{00} \approx 2R_{00} \approx 8\pi G\, T_{00}$. [Note: This G is Newton's gravitational constant, not $G_\mu{}^\mu$.]

Thus we are led to guess Einstein's equation:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

This is a relation between two tensors, so is covariant. Of course there is another tensor whose covariant divergence vanishes, $g_{\mu\nu}$, and we might have written

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2}$$

Λ is called the cosmological constant. It must be small because empty space (no matter) could not be flat, $G_{\mu\nu} = -\Lambda g_{\mu\nu}$ in empty space, and we therefore have had a limit $|\Lambda| < 10^{-56}\text{cm}^{-2}$, or $|\Lambda|^{-1/2} > 10^{10}$ lightyears. For calculations on motions $\ll 10^{10}$ lightyears, $|\Lambda|^{-1/2}$ might as well be ∞, $\Lambda = 0$.

When it comes to cosmology it matters whether $\Lambda = 0$ or not. Einstein originally did not include this term, but he found he could not find a stable configuration for the universe. So he postulated the existence of the cosmological term to make it possible for the universe to sit still. Of course, Hubble found that it wasn't sitting still at all, but blowing up since the big bang. Einstein called his postulating the cosmological constant "the biggest blunder of my life. But it wasn't a mistake — it may very well be the "dark energy" that has everyone so excited now.

2.1 Number of degrees of freedom

We have found the equation that determines $G_{\mu\nu}$ in terms of the matter fields (if $\Lambda = 0$, or a combination of $G_{\mu\nu}$ and $g_{\mu\nu}$ if $\Lambda \neq 0$), but this is only

a constraint on ten combinations of $R^\rho{}_{\sigma\mu\nu}$. How do we count the degrees of freedom which are undetermined?

For a scalar field, say one that satisfies the Klein-Gordon equation

$$\partial_\mu\phi\, \partial^\mu\phi + m^2\phi^2 = 0,$$

we see that on an initial surface $t = t_0$, one can choose $\phi(\vec{x}\,)$ and $\dot\phi(\vec{x}\,)$ independently at each spacial point, and its subsequent behavior is determined. In analogy to a point particle, where $x(t_0)$ and $\dot{x}(t_0)$ are required to determine its subsequent motion, we call this one degree of freedom for each point in space. Fourier transforming tells us

$$\tilde\phi = \int d^3k\, \tilde\phi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x} - i\omega(|k|)t},$$

with $\omega(|k|) = \sqrt{k^2 + m^2}$, with $\tilde\phi_{\vec{k}}$ an arbitrary function of three-dimensional momentum.

But counting degrees of freedom when there is a gauge invariance is more complicated. For example, a massive vector field $A^\mu(\mathbf{x})$ satisfies $\partial_\mu F^{\mu\nu} + m^2 A^\mu = 0$ (with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ as usual). This implies $\partial_\nu A^\nu = 0$ as well as the Klein-Gordon equation for all components A^ν. Thus

$$\tilde A^\mu = \int d^3k\, \tilde A^\mu_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x} - i\omega(|k|)t},$$

with $\omega(|k|) = \sqrt{k^2 + m^2}$, but with the constraint $\vec{k}\cdot\vec{A}_{\vec{k}} - \omega(|k|)A^0_{\vec{k}} = 0$. Thus the four components of A^μ actually only describe three degrees of freedom. And it gets worse if $m = 0$, because then, as we know, there is a gauge invariance that tells us not all of A^μ is physical, that one of these degrees of freedom is an arbitrary gauge transformation and not a physical degree of freedom.

Now for gravity, things are much worse, as we know we can make an arbitrary change in the chart coordinates at a point (a GL(4) with 16 arbitrary parameters). So there is a tremendous set of gauge-like transformations that do not correspond to physical degrees of freedom.

Let us count the independent degrees of g and R. g is a symmetric 4×4 matrix, so has 10 parameters. R has four indices which can each take on 4 values, but there are lots of constraints from symmetry. It is antisymmetric in the last two indices, so there are 6 possibilities there, and also, if we lower the first index, it is antisymmetric on those, so 6 possibilities there. But it is symmetric under interchange of the first two and the last two, so $\binom{6}{2} = 21$ possibilities. Finally there is one constraint from cyclicity, so finally R lives in a 20 dimensional space.

The symmetry of interchanging the first two with the last two indices also tells us that $R_{\mu\nu}$ and $G_{\mu\nu}$ are symmetric tensors, which is why I said 10 field equations.

But there are not 20, or even 10, free parameters for the physical gravitational field. In fact, there are just two degrees of freedom for each $\vec{k}$ wave, just as for the photon field. But the field is a tensor field rather than a vector, describing a massless spin 2 object. A gravitational wave is transverse, just as a photon is, but corresponds to elliptical distortions which can be along one transverse axis or along one 45 degrees rotated (around $\vec{k}$.)

2.2 Deriving the Gravitational Field Equations

Physics begins with an action:

$$S = \int d^4x \sqrt{g}\, \mathcal{L}.$$

$\mathcal{L}$ is a scalar Lagrangian density, a function of $g_{\mu\nu}$ and the matter degrees of freedom, that is, all other dynamical variables other than space-time. Divide it into

$$\mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{matter}}.$$

Here $\mathcal{L}_{\text{grav}}$ depends only on $g_{\mu\nu}$ and its derivatives, while $\mathcal{L}_{\text{matter}}$ is specified by extrapolation, using the equivalence principle, from a world where $R = 0$. So we expect $\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{matter}}(g_{\mu\nu}, \{\psi\})$ to not depend on derivatives of g.

What scalar can we take for $\mathcal{L}_{\text{grav}}$? The only one involving two derivatives of g is R. One could also add a constant Λ.

Euler's equations need to be reconsidered as R involves second derivatives, as well as the first derivatives squared that we are used to seeing in ordinary lagrangian mechanics. If we take

$$\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G} R - \frac{1}{8\pi G}\Lambda$$

and vary S with respect to $g_{\mu\nu}$, and insist on no variation of the action, we find the field equation Eq. (2).