Physics 464/511 Lecture P Fall, 2016

1 Equivalence Principle

I am anxious to get into general relativity. We will follow the motivation of Einstein, who was clearly led to his conception of general relativity by analogy with his success in special relativity. Let us examine the beginning of his first paper on relativity:

On the Electrodynamics of Moving Bodies
by A. Einstein

> It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the con- ductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.
>
> Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.

Now in special relativity we restrict our attention to inertial frames. But in 1911 Einstein realized that the same multiple explanation situation applied to considering the physics for a person in a big box, considering the

motion of objects in the box, whether the box was sitting on the surface of a big planet or out in gravity-free space but being accelerated. Consider the mechanics of a physicist $\mathcal{O}'$ in a closed room, which is accelerating at a constant acceleration a, and is therefore not an inertial frame. From the special relativity approach we situate ourselves, named $\mathcal{O}$, in an inertial frame with respect to which he ($\mathcal{O}'$) has velocity v at a given instant. If we restrict ourselves to an interval over which v is small, we find every object in his room obeys $\vec{F}_i = m\dfrac{d^2\vec{x}_i}{dt^2} = m\vec{a}_i$. Using his coordinates we find $\vec{a}\,'_i = \vec{a}_i - \vec{a}$, so

$$m\vec{a}\,'_i = \text{``}\vec{F}'_i\text{''} = m\vec{a}_i - m\vec{a} = \vec{F}_i - m\vec{a}.$$

If the observer in the box tries to use Newton's laws, he looks for the physical origin of the force $\vec{F}\,'_i$. But the objects which are interacting with the observed object generate only the force $\vec{F}_i$, and he must postulate a pseudoforce $-m\vec{a}$ due to no definable other object. If he wishes to conclude that he must be accelerating, he must exclude the possibility that this force is due to some other object from outside. Perhaps he reasons: all other forces depend on positions, charges, and other variables of the material. But this excess force is always proportional to the mass, exactly as it would be if I were accelerating. Therefore I conclude that there are no outside influences, but I am accelerating with respect to an inertial frame.

But would he not observe exactly the same physics within his box if it was simply sitting on the surface of a large planet? Each object within the box would experience an extra force mg downwards, so that the situation would be indestinguishable from a box accelerating with $a = g$ in the opposite direction.

Now you should argue that the way real forces are distinguished from pseudoforces is that they depend on some property of the object, such as charge, rather than being proportional to the inertial mass. Perhaps the gravitational mass in $W = m_g g$ is not exactly the same as the inertial mass m_I. Any relativity book will tell you of the ingeneous experiments which attempt to find a variation in $m_g/m_I = 1 + \delta$ and show that $|\delta| < 10^{-12}$. So the masses appear to be equal. This equality is so accurately known that it rules out possibilities like leaving out from m_g

- the binding energy of an atom, $\approx 10^{-8}$ in hydrogen
- the Lamb shift energy, 4×10^{-12} in hydrogen, more in other atoms.

So once again we have a situation with two different explanations of the same observations depending on coordinate system. Once again Einstein raised the equivalence under certain conditions to a fundamental postulate, called the **principle of equivalence**.

Before we get too carried away, we must examine more carefully what this equivalence is. In the box on the surface of the Earth, the objects do not really all accelerate the same, because different points are different distances away from the center of the Earth, and the accelerations are all pointing towards the center of the Earth, and are therefore not exactly parallel to each other. If the coordinates are x^i, we will find

$$\frac{d^2x^i}{d\tau^2} = a^i(x^j) = a^i(0) + x^j\,\partial_j a^i\big|_0 + \ldots$$

where 0 is within the box, and we will think of the box's extent (range of $\vec{x}$) as small compared to the variation scale of a (that is, $x \ll a/\partial_j a$). The $a^i(0)$ term is the same for all particles in the box, and can be considered a pseudoforce due to acceleration of the box. But the second term, which gives the variation of the accelerations, is a detectable effect, driving objects towards the floor and roof and in from the sides of a satellite in free fall. These are called **tidal forces**. So we cannot say that all gravitational forces are pseudoforces, but only that the gravitational force at any particular point may be considered a pseudoforce.

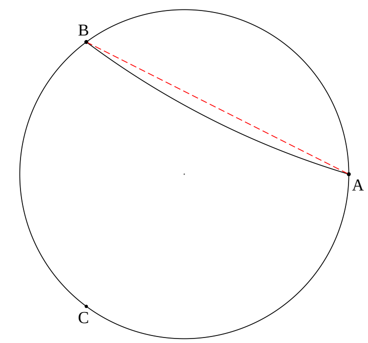
In the absence of gravity, the equations of motion are given by the laws of special relativity, together with whatever the relevant mechanics of the matter is. By the equivalence principle, if we can set up a coordinate system in which there are no gravitational forces, then physics obeys special relativistic laws in that coordinate system. In other coordinate systems, we must expect physics to be weird.

We all know that if you try to describe mechanics from an accelerating frame there are strange forces. For example, in a rotating system there are centrifugal and Coriolus forces. But there is worse.

Consider[1] a rotating table, and let observers moving with the table at-

[1]Reference: Feynman, Lectures in Physics II, chapter 42.

tempt to draw a triangle. They draw straight lines from A to B, *etc.*. What does straight line mean? The shortest distance between two points. So they draw two paths as shown. The dashed line looks straight to us, but when the residents of the turntable go to compare the lengths, they find it is longer than the one we consider curved. Why? According to us, not rotating, their metersticks shrink increasingly as they go away from the center, especially when held tangentially, so they are measuring the dashed line with shrunken metersticks, and more of them fit along that line than along the one that appears curved to us. If they do the same between B and C, and between C and A, and measure the angles, they will find the sum of the angles of their triangle is less[2] than 180°! Actually more convincing, the bugs measure the same radius R that we do, but their measurement of the circumference, which we claim is $2\pi R$, with their metersticks shrunking to $1/\gamma$ meters, is $2\pi\gamma R > 2\pi R$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$ with v the velocity of the rim. Geometry is not Euclidian or Minkowskian when observed in an accelerating coordinate system.



Let us return to our box which may be accelerating through empty space or may be sitting on the surface of a large planet, with no way for us to tell which. A photon comes through a one-way window and crosses the box. If we are an accelerating spaceship, an inertial observer looking in sees the photon moving in a straight line, as would any other free particle, while our box accelerates upwards with acceleration g. Therefore to an observer with coordinates fixed in the box, the photon falls with the same acceleration g as all other particles. This requires that light is bent in a gravitational field, so that, for example, star light passing the sun should be bent inwards, and stars observed on opposite sides of the sun during a solar eclipse should appear to be further apart than usual. We may return to this later, as if we did the calculation now we would get the wrong answer by a factor of two.

Another conclusion we may reach is even more startling, though not quite so simple. Suppose we have two clocks, one at the top of the spaceship-box

[2]Actually, this is not clear, because what looks like less than 60° to us will look bigger to them, as the meterstick used to measure the separation will shrink.

and one at the bottom, a distance h apart. Let us observe with an inertial observer $\mathcal{O}$, at a time when the velocity of the ship is small. If the bottom clock emits a flash of light when $v = 0$, it will not be received by the top clock until time $h/c = h$, at which time the clock will be moving away from the source at $v = ah$. The light will therefore be red-shifted by

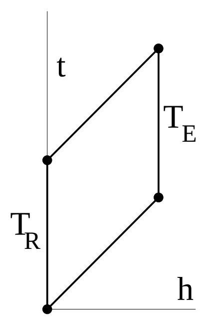
$$f_{\rm top} = f_{\rm bottom}\sqrt{\frac{1-ah}{1+ah}} \approx f_{\rm bottom}\,(1-ah)\,.$$

Similarly if the clock on top emits a flash when $v = 0$, the bottom one will receive it at time h, at which time it is moving towards the source at velocity ah, and the light is blue-shifted

$$f_{\rm bottom} = f_{\rm top}\sqrt{\frac{1+ah}{1-ah}} \approx f_{\rm top}\,(1+ah)\,.$$

This agrees with the previous equation, and both observers agree that the frequency of ticks of the bottom clock is lower than that of the top, or the higher clock is running faster!

Now suppose our box is not a spaceship but the Empire State Building. Einstein says physics is the same, and the executives at the top are ageing faster than the receptionist on the first floor, at a rate $1+gh = 1+gh/c^2$ faster, which makes them about 1 μs older for each year they worked. Although this effect is probably not the correct explanation of their gray hair, it does lead us to an interesting conclusion: spacetime as measured on a planet's surface is not Minkowskian!. If the receptionist emits light rays one second apart, each travels up the Minkowski diagram at 45°, forming a parallelogram, but $T_E > T_R$.



This was presaged by our discussion of the turntable: accelerated observers do not see Minkowskian geometry. Any hope for Minkowskian geometry can only be for an inertial observer who feels no gravity. Given any particular event we can always find such an observer by letting him free-fall, but in his coordinate system gravity vanishes only in the neighborhood of the chosen event. There is no way to set up a global coordinate system which in inertial, so there is no way to treat the global geometry as Minkowskian. We are going to have to learn how to talk about curved spacetime.

We have seen that the spacetime in which physics acts is a curved space which can be considered flat (Minkowskian) in a small neighborhood at each point but cannot be considered flat globally[3]. In each region, I can find a coordinate system x^μ which is in 1−1 correspondence with the spacetime in that region. Such a 1−1 map from a region of spacetime to an open subset of $\mathbb{R}^4$ is called a chart. There is not necessarily a single chart which can cover the whole spacetime. We discussed setting up charts for a manifold in Lectures D and E, where we considered differentiation in terms of n-forms, learned how to integrate those and how they corresponded to certain vectors and tensors in flat space, but we also considered Riemannian manifolds with a nonconstant metric $g_{\mu\nu}$ and geodesics on such manifolds.

2 Vierbeins, Connections

[Ref: Weinberg *Gravitation and Cosmology* Part 2 Chapter 3]

Physics is described locally by fields, forms, and the metric tensor. At any point, the principle of equivalence tells us it is possible to choose a Minkowskian coordinate system with $g = \eta$, with no gravitational forces at that point. Let us set up a chart with coordinates ξ^α near the point $\mathcal{P}$ which is Minkowskian in the following sense:

- A free object at $\mathcal{P}$ has no acceleration in terms of the ξ coordinates, $d^2\xi^\alpha/d\tau^2 = 0$
- $\mathbf{g} = \mathbf{d}\xi^\alpha \otimes \mathbf{d}\xi^\beta \eta_{\alpha\beta}$ at $\mathcal{P}$.

[Note: the coordinates ξ^α are specially chosen to match the point $\mathcal{P}$, and more properly should be called $\xi^\alpha_{\mathcal{P}}$.] Einstein assures us that we can write down physics locally, at $\mathcal{P}$, in the coordinate system ξ, and it is the same as it would be were their no gravity.

The coordinates $\xi^\alpha_{\mathcal{P}}(\mathcal{P}')$ of the point $\mathcal{P}'$ have no decent properties except for $\mathcal{P}'$ at or near $\mathcal{P}$. In fact, we could have chosen a new chart $\xi^\alpha_{\mathcal{P}'}$ centered at $\mathcal{P}'$ to have things look Minkowskian there. Let us simultaneously use another chart $C = \{x^\mu\}$. Then

$$\mathbf{d}\xi^\alpha = V^\alpha_{\ \mu}\mathbf{d}x^\mu, \quad \text{where} \quad V^\alpha_{\ \mu}(\mathcal{P}) = \left.\frac{\partial \xi^\alpha}{\partial x^\mu}\right|_{\mathcal{P}}.$$

[3]Refs: more formal— Chapter 2 of Hawking and Ellis, *Large Scale Structure of Space-Time.* Less formal— Misner Thorne and Wheeler *Gravitation*, chapter 2.

The object $V^\alpha_{\ \mu}(\mathcal{P})$ is called the **Vierbein**.

The components of $\mathbf{g}(\mathcal{P})$ in C are

$$\mathbf{g} = g_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu = \eta_{\alpha\beta}\mathbf{d}\xi^\alpha \otimes \mathbf{d}\xi^\beta = \eta_{\alpha\beta}V^\alpha_{\ \mu}V^\beta_{\ \nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$$

$$\text{so} \qquad g_{\mu\nu} = \eta_{\alpha\beta}V^\alpha_{\ \mu}V^\beta_{\ \nu}\ .$$

The vierbein therefore determines the metric tensor.

What is the equation of motion? If there are no non-gravitational forces,

$$\frac{d}{d\tau}\frac{d\xi^\alpha}{d\tau} = 0 = \frac{d}{d\tau}\left(V^\alpha_{\ \mu}\frac{dx^\mu}{d\tau}\right) = V^\alpha_{\ \mu,\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + V^\alpha_{\ \mu}\frac{d^2x^\mu}{d\tau^2} = 0$$

V is the Jacobian of a nonsingular change of variables[4]. Its inverse is therefore

$$\left(V^{-1}\right)^\mu_{\ \alpha} = \frac{\partial x^\mu}{\partial \xi^\alpha}, \quad \text{as} \quad \left(V^{-1}\right)^\mu_{\ \alpha}V^\alpha_{\ \nu} = \frac{\partial x^\mu}{\partial \xi^\alpha}\cdot\frac{\partial \xi^\alpha}{\partial x^\nu} = \delta^\mu_\nu.$$

$$\text{Thus} \qquad \frac{d^2x^\rho}{d\tau^2} + \underbrace{\left(V^{-1}\right)^\rho_{\ \alpha}V^\alpha_{\ \mu,\nu}}_{\Gamma^\rho_{\ \mu\nu}}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

where we have defined the **affine connection**

$$\Gamma^\rho_{\ \mu\nu} := \left(V^{-1}\right)^\rho_{\ \alpha}V^\alpha_{\ \mu,\nu} = \frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu} \tag{1}$$

Thus we have the equation of motion

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\ \mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0 \tag{2}$$

This is also known as the geodesic equation, not only in general relativity but also on a Riemannian manifold. We saw this earlier in Lecture D.

Let us examine the relation of the affine connection to the metric. Note that as $\Gamma^\rho_{\ \mu\nu} := \left(V^{-1}\right)^\rho_{\ \alpha}V^\alpha_{\ \mu,\nu}$, $\quad V^\alpha_{\ \mu,\nu} = V^\alpha_{\ \rho}\Gamma^\rho_{\ \mu\nu}$, so

$$\begin{aligned} g_{\mu\nu,\rho} &= \frac{\partial}{\partial x^\rho}\left(V^\alpha_{\ \mu}V^\beta_{\ \nu}\eta_{\alpha\beta}\right) = \left(V^\alpha_{\ \mu,\rho}V^\beta_{\ \nu} + V^\alpha_{\ \mu}V^\beta_{\ \nu,\rho}\right)\eta_{\alpha\beta} \\ &= \left(\Gamma^\sigma_{\ \mu\rho}V^\alpha_{\ \sigma}V^\beta_{\ \nu} + \Gamma^\sigma_{\ \nu\rho}V^\alpha_{\ \mu}V^\beta_{\ \sigma}\right)\eta_{\alpha\beta} = \Gamma^\sigma_{\ \mu\rho}g_{\sigma\nu} + \Gamma^\sigma_{\ \nu\rho}g_{\sigma\mu} \end{aligned}$$

[4]Notation I should have introduced earlier: a subscript $\{\},\mu$ means derivative of whatever is $\{\}$ with respect to x^μ, so $V^\alpha_{\ \mu,\nu} := \frac{\partial V^\alpha_{\ \mu}}{\partial x^\nu}$.

Note we have assumed $\eta_{\alpha\beta,\rho} = 0$! So ξ is more than just an orthonormal set of coordinates at $\mathcal{P}$, it is also one with no acceleration without forces.

The vierbein is not a tensor, because it refers to two different charts. Γ has only indices which refer to the chart C, but nonetheless it is not a tensor. We shall see later how it changes under chart change. Nonetheless, let us raise and lower its indices with g, so

$$g_{\mu\nu,\rho} = \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho}, \quad \text{but also} \quad \Gamma_{\sigma\mu\nu} = \Gamma_{\sigma\nu\mu}.$$

Add the same with $\mu \leftrightarrow \rho$ and subtract $\nu \leftrightarrow \rho$,

$$\begin{aligned} g_{\rho\nu,\mu} &= \Gamma_{\nu\rho\mu} + \Gamma_{\rho\nu\mu} = \Gamma_{\nu\mu\rho} + \Gamma_{\rho\nu\mu} \\ -g_{\mu\rho,\nu} &= -\Gamma_{\rho\mu\nu} - \Gamma_{\mu\rho\nu} = -\Gamma_{\rho\nu\mu} - \Gamma_{\mu\nu\rho} \end{aligned}$$

so, adding and dividing by two,

$$\frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right) = \Gamma_{\nu\mu\rho}$$

and $\Gamma^\sigma_{\ \mu\rho} = g^{\sigma\nu}\,\Gamma_{\nu\mu\rho} = \dfrac{1}{2}\,g^{\sigma\nu}\left(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right)$.

Having derived the equation which determines how otherwise free particles move in a gravitational field, let us compare with Newton's laws for a test particle in the field of a single heavy body moving non-relativistically. We must limit our attention to slow particles and weak gravitational fields, for otherwise Newton can't be right. Furthermore, it should be possible to choose our coordinates so that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$, so $\Gamma \ll 1$. Then to first order in h, Γ and v, $t = \tau$, $u^\mu = (1, \vec{v})$, and the geodesic equation and $\vec{F} = m\vec{a} = -\vec{\nabla}\phi$ give

$$\frac{d^2x^j}{dt^2} = -\Gamma^j_{\ 00} = -\partial_j\phi,$$

where ϕ is Newton's gravitional potential $\phi = -GM/r$. Assume $g_{\mu\nu}$ is independent of time. Then

$$\Gamma^j_{\ 00} \approx \frac{1}{2}\left(g_{j0,0} + g_{0j,0} - g_{00,j}\right) = -\frac{1}{2}g_{00,j}$$

so $g_{00} = 1 - 2\phi$. This is the Newtonian approximation.

Consider now a stationary metric, $g_{\mu\nu}(\vec{x})$ independent of t, not necessarily weak. Consider two clocks at rest in this field. Each clock is guaranteed by the manufacturer to tick once each second of proper time regardless of acceleration (no grandfather clocks allowed). In terms of our coordinate system

$$(\Delta\tau)^2 = -g_{\mu\nu}\Delta x^\mu\,\Delta x^\nu = -g_{00}(\vec{x})\;(\Delta t)^2$$

so the coordinate interval between ticks is

$$(\Delta t)_A = [-g_{00}(x_A)]^{-1/2}\,, \qquad (\Delta t)_B = [-g_{00}(x_B)]^{-1/2}\,.$$

If A sends light signals to B each time his clock ticks, the time differences $t_{\rm received} - t_{\rm emitted}$ will be the same for each pulse, so B can measure on his own clock the period between ticks of A's clock. The answer is

$$T' = \frac{\Delta t_A}{\Delta t_B} = \left[\frac{g_{00}(x_B)}{g_{00}(x_A)}\right]^{1/2}$$

and the frequency of the light emitted is therefore shifted by

$$f' = f\left[\frac{g_{00}(x_A)}{g_{00}(x_B)}\right]^{1/2}.$$

We have derived this for an arbitrary stationary metric. In the Newtonian limit

$$\frac{f'}{f} = \left[\frac{1+2\phi_A}{1+2\phi_B}\right]^{\frac{1}{2}} \approx 1 + (\phi_A - \phi_B) = 1 + GM\left[\frac{1}{r_B} - \frac{1}{r_A}\right].$$

At the surface of the Sun $\phi = -2.12\times 10^{-6}$, so for an observer at ∞, the Sun's light is red shifted by

$$\Delta f/f = +\phi_{\rm surface} = -2.12\times 10^{-6}.$$

Note that $f'/f = 1+\phi_A-\phi_B$ agrees with our calculation based on equivalence to a rocket ship.

This gravitational red shift is best tested by dropping photons down a shaft at Harvard. General relativity has been tested thereby to an accuracy of about 1%.