Physics 464 Lecture M Fall, 2016

1 Integral Transforms

An integral transform is a linear map $f \to g$ from a space of functions into another given by

$$g(\alpha) = \int_a^b f(t) K(\alpha, t) dt$$

where K is a fixed function called the **kernel**, and a and b are fixed. We write this map $\mathcal{L} : f \to g$ or $g = \mathcal{L} f$, or less correctly but more transparently,

$$g(\alpha) = \mathcal{L} f(t).$$

$\mathcal{L}$ is obviously linear. If the map is invertible, we call its inverse $\mathcal{L}^{-1}$. The question of which spaces of functions are to be considered for f and g can lead to murky mathematics, which we abhor.

Useful transforms:

$K(\alpha, t)$	(a, b)	Name
$\frac{1}{\sqrt{2\pi}} e^{i\alpha t}$	$(-\infty, \infty)$	Fourier
$e^{-\alpha t}$	$(0, \infty)$	Laplace
$t J_n(\alpha t)$	$(0, \infty)$	Hankel
$t^{\alpha - 1}$	$(0, \infty)$	Mellon

We will cover the fourier integral in terms of a limit of the fourier series.

If a function $f(x)$ is periodic on $[-L, L]$, we may change the interval with $u = \pi x / L$, and fourier transform the function $F(u) := f(x)$. Then F is periodic on $[-\pi, \pi]$ so

$$F(u) = \sum_{n=-\infty}^{\infty} a_n e^{inu}, \qquad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(u) e^{-inu}\, du$$

$$\text{or} \quad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L}, \qquad a_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-in\pi t/L}\, dt$$

$$\text{and} \quad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L} = \int_{-L}^{L} f(t) \sum_{n=-\infty}^{\infty} e^{in\pi(x-t)/L} \frac{dt}{2L}.$$

Let $\omega = n\pi/L$. Then summing on n means summing on ω's with a spacing $\Delta\omega = \pi/L$,

$$f(x) = \int_{-L}^{L} f(t) \left\{ \frac{1}{2\pi} \sum_{\omega} e^{i\omega(x-t)} \Delta\omega \right\} dt$$

We see that the $\{\,\}$ acts like a $\delta(x-t)$ on the interval $[-L, L]$. As $L \to \infty$, the sum becomes an integral,

$$f(x) = \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(x-t)} \right\} dt$$

$$\text{so} \quad \delta(x-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(x-t)}.$$

For any finite L, the function

$$\begin{aligned} g(\omega) &= L\sqrt{\frac{2}{\pi}}\, a_n \qquad \text{for } \omega = n\pi/L \\ &= \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(t)\, e^{-i\omega t}\, dt \end{aligned}$$

is defined only at a finite set of points, but as $L \to \infty$,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \tag{1}$$

is defined for all ω. The inverse is given by the expansion of f,

$$\begin{aligned} f(x) &= \sum a_n e^{in\pi x/L} = \sum_{\omega} \sqrt{\frac{\pi}{2}}\, \frac{1}{L}\, g(\omega)\, e^{i\omega x} = \frac{1}{\sqrt{2\pi}} \sum_{\omega} \Delta\omega\, g(\omega)\, e^{i\omega x} \\ &\to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, g(\omega)\, e^{i\omega x} \end{aligned} \tag{2}$$

(1) is called the **fourier transform** and (2) the **inverse fourier transform**. The sign in the exponentials are reversed from Arfken — this is a question of convention, as long as the sign of (1) is the opposite of that for (2).

It is conventional to call ω the variable for the fourier transform of a function of time t, and name it the *angular frequency*, and use k for the variable for the transform of a function of position x, and call it the wave-number. Then the conventional choices of signs for the exponents are

$$g(k,\omega) = \frac{1}{2\pi}\int dx\, dt\, f(x,t) e^{-ikx+i\omega t}$$

with different choices of sign for x and t, to account for the relativistic dot product $k_\mu x^\mu = \vec{k}\cdot\vec{x} - \omega t$.

The nice thing about fourier transforms is that derivatives become simple. If $h(x)$ is the fourier transform of df/dx,

$$\begin{aligned} \text{where} \qquad f(x) &= \frac{1}{\sqrt{2\pi}}\int dk\, e^{ikx} g(k) \\ \text{then} \qquad h(x) = \frac{df}{dx} &= \frac{1}{\sqrt{2\pi}}\int dk\, \{ikg(k)\}\, e^{ikx} \\ \text{so} \qquad h(k) &= ikg(k) \end{aligned}$$

Thus derivatives with respect to x turn into multiplication by ik. On the other hand multiplication of functions of x become difficult, turning into a convolution

$$\begin{aligned} \mathcal{L}\big(f(x)h(x)\big) &= \frac{1}{\sqrt{2\pi}}\int dx\, e^{-ikx}\, f(x)\, h(x) \\ &= \frac{1}{\sqrt{2\pi}}\int dx\, e^{-ikx} \int \frac{dq}{\sqrt{2\pi}} e^{iqx} \left(\mathcal{L}f\right)(q) \int \frac{dp}{\sqrt{2\pi}} e^{ipx} \left(\mathcal{L}h\right)(p) \\ &= \frac{1}{\sqrt{2\pi}}\int dq\, dp\, \delta(k-p-q) \big[\left(\mathcal{L}f\right)(q)\big]\cdot\big[\left(\mathcal{L}h\right)(p)\big] \\ &= \frac{1}{\sqrt{2\pi}}\int dq\, \big[\left(\mathcal{L}f\right)(q)\big]\cdot\big[\left(\mathcal{L}h\right)(k-q)\big]. \end{aligned}$$

This combination of $\mathcal{L}f$ and $\mathcal{L}h$ is called the **convolution**, and written $\mathcal{L}f * \mathcal{L}h$, so we have shown

$$\mathcal{L}(fh) = \mathcal{L}f * \mathcal{L}h.$$

Parseval was a medieval knight[1] who went in search of the holy grail, at

[1] Just kidding. That was Parsival or Perceval or Parzival.

least according to Arthurian legends. What he found was

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk\; (\mathcal{L}f(k))\; [\mathcal{L}g(k)]^* \\
&\quad = \int_{-\infty}^{\infty} dk \int \frac{dy}{\sqrt{2\pi}}\, e^{-iky}\, f(y) \left[\int \frac{dx}{\sqrt{2\pi}}\, e^{-ikx}\, g(x)\right]^* \\
&= \int dx\, dy\, f(y)\, g^*(x) \underbrace{\int \frac{dk}{2\pi} e^{ik(x-y)}}_{\delta(x-y)} = \int dx\, f(x)\, g^*(x),
\end{aligned}$$

so, with norm given by an integral with weight $w = 1$ for both the function space and the "dual" space of the fourier transform,

$$(g, f) = (\mathcal{L}g, \mathcal{L}f)\,.$$

I do not have time to discuss Laplace and Mellin transforms. Laplace transforms are used extensively in analysis of circuits.

Consider a general integral transform

$$f(x) = \int_a^b K(x,t)\; \phi(t)\, dt.$$

Then there is the problem of inverting, *i.e.* given f, to find ϕ. Considered thusly this is called a Fredholm equation. This form is called the first kind, while

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi(t) dt$$

is called a Fredholm equation of the second kind.

Formally solving these equations is like solving a linear equation, but the coefficients are operators.

1.1 Green's Functions

Let $\mathcal{L}$ be a self-adjoint differential operator of second order, $\mathcal{L} = \dfrac{d}{dx} p(x) \dfrac{d}{dx} + q(x)$, and suppose that we wish to solve the equation $\mathcal{L}\, y(x) + f(x) = 0$ for some known source function $f(x)$.

Let us assume that the solution y is given by

$$y(x) = \int G(x,t) f(t)\, dt$$

for some kernel G independent of y and f. This is reasonable because of the linear nature of y's dependence on f.

Applying $\mathcal{L}$,

$$\mathcal{L}y = \int \mathcal{L}_x G(x,t)\, f(t)\, dt = -f(x),$$
$$\text{so} \qquad \mathcal{L}_x G(x,t) = -\delta(x-t).$$

Thus $G(x,t)$ is a solution of the homogeneous equation $\mathcal{L}_x G(x,t) = 0$, except at the point $x = t$. Then

$$\int_{t-\Delta x/2}^{t+\Delta x/2} \mathcal{L}_x G(x,t)\, dx = -\int_{t-\Delta x/2}^{t+\Delta x/2} \delta(x-t)\, dx = -1$$
$$= p(x)\, \frac{d}{dx} G(x,t)\Big|_{t-\Delta x/2}^{t+\Delta x/2} + \int_{t-\Delta x/2}^{t+\Delta x/2} q(x)\, G(x,t)\, dx.$$

If we assume G is bounded, the second term vanishes as $\Delta x \to 0$ and we get a discontinuity in dG/dx,

$$p(t)\, \frac{\partial G}{\partial x}\Big|_{x=t+\epsilon} = p(t)\, \frac{\partial G}{\partial x}\Big|_{x=t-\epsilon} - 1.$$

We will impose the condition that G itself be continuous.

Suppose we expect our solution y to obey some homogeneous boundary condition at a and b, such as $y(a) = 0$. Then the solution $y(x)$ will satisfy this condition if $G(x,t)$ does, *e.g.* $G(a,t) = 0$. Let $u(x)$ be the solution of $\mathcal{L}u = 0$ on $[a,t]$ and $v(x)$ the solution on $[t,b]$, satisfying the specified boundary conditions.[2] Each is arbitrary up to a multiplicative constant. So

$$G(x,t) = \begin{cases} c_1 u(x) & \text{for } x < t \\ c_2 v(x) & \text{for } t < x \end{cases}$$

[2]Generically u and v cannot each satisfy both homogeneous conditions without vanishing, so u and v will not be the same function. This differs from the Sturm-Liouville situation, where the equation had an adjustible parameter λ for discrete values of which $u(x)$ could satisfy both boundary conditions.

satisfies our conditions if

$$\begin{aligned} c_1 u(t) - c_2 v(t) &= 0 \\ c_1 u'(t) - c_2 v'(t) &= 1/p(t) \end{aligned}$$

which has a solution for c_1 and c_2 if the Wronskian

$$\left.\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}\right|_{x=t} \neq 0.$$

If we choose u and v fixed functions independent of the t we are dealing with, the t dependence enters through $c_1(t)$ and $c_2(t)$. Then

$$\begin{aligned} \frac{d}{dx}\left(p(x)\begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}\right) &= \frac{d}{dx}\left\{u(x)p(x)\frac{dv}{dx} - v(x)p(x)\frac{du}{dx}\right\} \\ &= \frac{du}{dx}p(x)\frac{dv}{dx} + u(x)(-q(x))v(x) - \frac{dv}{dx}p(x)\frac{du}{dx} - v(x)(-q(x))u(x) = 0, \end{aligned}$$

because $\mathcal{L}u = \mathcal{L}v = 0$. So $p(x)\left(u(x)v'(x) - v(x)u'(x)\right) = A$, a constant, and $c_1(t) = -\dfrac{v(t)}{A}$, $c_2(t) = -\dfrac{u(t)}{A}$ clearly satisfy our equations. Thus

$$G(x,t) = \begin{cases} -\dfrac{1}{A}u(x)v(t) & \text{for } x < t \\ -\dfrac{1}{A}u(t)v(x) & \text{for } x > t \end{cases}$$

satisfies all our conditions. If we can find u and v we can find the Green's function G and then the solution to the inhomogeneous equation

$$\mathcal{L}\, y(x) + f(x) = 0.$$

In Lecture I we considered the homogeneous self-adjoint second order differential equation

$$\frac{d}{dx}p(x)\frac{dy(x)}{dx} + q(x)y(x) + \lambda w(x)y(x) = 0,$$

on an interval $[a,b]$ with suitable homogeneous boundary conditions, and found there were nonzero solutions ϕ_n only for a discrete infinite set of eigenvalues λ_n, and that these eigenfunctions could be normalized such that they were orthogonal with weight $w(x)$, in the sense that

$$\langle \phi_r, \phi_s \rangle := \int_a^b w(x)\phi_r^*(x)\phi_s(x)dx = \delta_{rs},$$

and that they satisfied

$$\sum_r w(t)\phi_r^*(t)\phi_r(x) = \delta(x-t).$$

Now consider adding a source term, looking for a solution to

$$\frac{d}{dx}p(x)\frac{dy(x)}{dx} + q(x)y(x) + \lambda w(x)y(x) = f(x)$$

for a given function f. Again, we expect there to be a Green's function, now depending on the parameter λ as well as x and t, with

$$y(x) = \int_a^b G_\lambda(x,t)f(t)dt.$$

With the hermitean operator defined by $\mathcal{L} := \dfrac{d}{dx}p(x)\dfrac{d}{dx} + q(x)$, our eigenfunctions satisfy $\mathcal{L}\phi_n(x) = -\lambda_n w(x)\phi_n(x)$ and the Green's function satisfies $[\mathcal{L}_x + \lambda w(x)]\, G_\lambda(x,y) = -\delta(x-y)$. We saw at the end of Lecture I that

$$G_\lambda(x,y) = \sum_n \frac{\phi_n(x)\phi_n^*(y)}{\lambda - \lambda_n}.$$

Green's functions are very useful in partial differential equations, as long as they are linear. Consider again

$$\mathcal{L}\,\phi(\vec{r}\,) + f(\vec{r}\,) = 0$$

where now $\mathcal{L}$ is a partial differential operator on a function $\phi(\vec{r}\,)$. We again define the Green's function by

$$\mathcal{L}_{r_1}\left(G(\vec{r}_1,\vec{r}_2)\right) = -\delta^D(\vec{r}_1 - \vec{r}_2)$$

plus some boundary conditions when $\vec{r}_1$ is on the surface of the region being considered (perhaps $|\vec{r}_1| \to \infty$). Then

$$y(\vec{r}_1) = \int d^D\vec{r}_2\; G(\vec{r}_1,\vec{r}_2)f(\vec{r}_2)$$

satisfies $\mathcal{L}y(\vec{r}_1) = \int d^D\vec{r}_2\,\{\mathcal{L}_{r_1}G(\vec{r}_1,\vec{r}_2)\}\,f(\vec{r}_2) = -\int d^D\vec{r}_2\;\delta^D(\vec{r}_1-\vec{r}_2)f(\vec{r}_2) = -f(\vec{r}_1)$.

The most important cases are, of course the Laplacian $\mathcal{L} = \nabla^2$ and the Helmholtz $\mathcal{L} = \nabla^2 + k^2$. As the $\mathcal{L}$ operator is translation invariant, $G(\vec{r}_1, \vec{r}_2) = G(\vec{r}_1 - \vec{a}, \vec{r}_2 - \vec{a})$ which means it is a function only of $\vec{r}_1 - \vec{r}_2$. In addition, $\mathcal{L}$ is rotation invariant, so it is a function only of $|\vec{r}_1 - \vec{r}_2|$,

$$G(\vec{r}_1, \vec{r}_2) = F(|\vec{r}_1 - \vec{r}_2|).$$

To evaluate F, we can set $\vec{r}_2 = 0$.

We can consider the Laplacian Green's function as the $k = 0$ special case of Helmholtz. Consider a sphere $|\vec{r}_1| = R$ and ask what the integral over that sphere of $\hat{n} \cdot \vec{\nabla} G(\vec{r}_1, 0) = \int d^2\Omega R^2 F'(R)$ is, where $F'(R) = dF/dr|_R$. By Gauss's law, this

$$\begin{aligned} 4\pi R^2 F'(R) &= \int_V \vec{\nabla} \cdot \vec{\nabla} G(\vec{r}, 0) = \int_V \left(\mathcal{L}_{r_1} - k^2\right) G(\vec{r}, 0) \\ &= -1 - k^2 \int_V G(\vec{r}, 0) = -1 - 4\pi k^2 \int_0^R r^2 F(r)\, dr. \qquad (3) \end{aligned}$$

Differentiating in R gives $2RF'(R) + R^2 F''(R) + k^2 R^2 F(R) = 0$, which is the spherical Bessel equation with $\ell = 0$, with solutions $j_0(kR) = \sin(kR)/kR$ and $n_0(kR) = -\cos(kR)/kR$. Thus $rF(r) = Ae^{ikr} + Be^{-ikr}$, the left hand side of (3) is $4\pi \left[ikR(Ae^{ikR} - Be^{-ikR}) - Ae^{ikR} - Be^{-ikR}\right]$ and the right hand side is

$$\begin{aligned} &-1 - 4\pi k^2 \int_0^R (Are^{ikr} + Bre^{-ikr})\, dr \\ &\quad = -1 + 4\pi A \left[(ikR - 1)e^{ikR} + 1\right] - 4\pi B \left[(ikR + 1)e^{-ikR} - 1\right] \end{aligned}$$

which agrees as long as $A + B = \dfrac{1}{4\pi}$. If we choose $B = 0$, we have

$$G(\vec{r}_1, \vec{r}_2) = \frac{e^{ik|\vec{r}_1 - \vec{r}_2|}}{4\pi|\vec{r}_1 - \vec{r}_2|}.$$

What do we make of the ambiguity? Adding $\frac{B}{4\pi r}(e^{-ikr} - e^{ikr}) = \frac{B}{2\pi i r}\sin(kr)$ is adding a solution of the homogeneous equation $\mathcal{L}\Delta G = 0$, so, as usual, the solution of the inhomogeneous equation is ambiguous by addition of the homogeneous one. Boundary conditions on the solution at infinity should determine that ambiguity.

Returning to the Poisson equation with $\mathcal{L}$ just the laplacian, we have the Green's function

$$G(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi} \frac{1}{|\vec{r}_1 - \vec{r}_2|}.$$

In spaces other than three dimensional, we would still have, by translational invariance and rotational symmetry, $G(r_1, r_2) = f(r)$, so on a hypersphere of radius R about r_2, $\vec{\nabla} G = f'(R)\hat{e}_r$, where $\vec{r} = \vec{r}_1 - \vec{r}_2$. Then Gauss' law would tell us

$$\begin{aligned}
\int_{|r|=R} \hat{n} \cdot \vec{\nabla} G &= \int_{|r|\le R} \nabla^2 G = \int_{|r|\le R} \big(\mathcal{L}G - k^2 G\big) \\
&\qquad = \int_{|r|\le R} \big(-\delta^D(\vec{r}) - k^2 f(|r|)\big) = -1 - S_D k^2 \int_0^R r^{D-1} f(r)\, dr \\
&= S_D R^{D-1} f'(R)
\end{aligned}$$

where $S_D = \int d\Omega_D$ is the surface area of a unit ball in D dimensions, or the $D{-}1$ sphere.

Differentiating with respect to R we find

$$r^{D-1} f'' + (D{-}1) r^{D-2} f' + k^2 r^{D-1} f = 0,$$

Let's again restrict ourselves to the Poisson equation with $k = 0$. Then we have $S_D R^{D-1} f'(R) = -1$, and $G(\vec{r}_1, \vec{0}) = \frac{-1}{S_D} \int_0^{|r_1|} r^{1-D} dr = \frac{1}{(D-2)\, S_D} |r_1|^{2-D}$. or

$$G(\vec{r}_1, \vec{r}_2) = \frac{1}{(D{-}2)\, S_D\, |\vec{r}_1 - \vec{r}_2|^{D-2}}.$$

You know $S_2 = 2\pi$ and $S_3 = 4\pi$. As is shown in the supplementary notes[3] $S_D = 2\frac{\pi^{D/2}}{\Gamma(D/2)}$, thus we have

$$G(\vec{r}_1 - \vec{r}_2) = \frac{\Gamma(D/2)}{(D{-}2)} \frac{1}{2\pi^{D/2} |\vec{r}_1 - \vec{r}_2|^{D-2}} = \frac{\Gamma(\frac{D-2}{2})}{4\pi^{D/2} |\vec{r}_1 - \vec{r}_2|^{D-2}} \qquad (4)$$

In particular, for $D = 3$, we get $f = \Gamma(1/2)/4\pi^{3/2}|r| = 1/4\pi r$, which we could also get with

$$A = 4\pi R^2, \quad f' = -\frac{1}{4\pi R^2}, \quad f = \frac{1}{4\pi R}$$

in agreement with Coulomb's law.

For $D = 2$, $A = 2\pi R, \quad f' = -\dfrac{1}{2\pi R}, \quad f = -\dfrac{1}{2\pi} \ln R.$

[3] "$\Gamma(N/2)$ and the Volume of S^{D-1}".

Trying to evaluate f with Eq. (4) is tricky, as it seems G goes to an infinite constant as $D \to 2$. Of course a solution to $\nabla^2\phi = -\rho$ has an undetermined constant in $\rho(\vec{r})$, so a constant in $G(\vec{r})$ is ignorable. Expanding $r^{2-D} = \exp\{(2-D)\ln r\} = 1 + (2-D)\ln r + \mathcal{O}(D-2)^2$ and dropping the 1, L'Hôpital's rule gives

$$G(\vec{r}_1, \vec{r}_2) = \lim_{D\to 2} \frac{1}{4\pi^{D/2}}\,\Gamma\left(\frac{D-2}{2}\right)(2-D)\ln r = -\frac{1}{2\pi}\ln r.$$

Note that in two dimensions the potential blows up at infinity. Thus Coulomb forces are confining in ≤ 2 dimensions.