Physics 464 Lecture L Fall, 2016

1 Fourier Analysis

Consider the self-adjoint (and simple) equation

$$\frac{d^2y}{d\phi^2} + n^2 y = 0$$

on the interval $[-\pi, \pi]$ with boundary conditions $y(-\pi) = y(\pi)$, $y'(-\pi) = y'(\pi)$. The solutions are $e^{\pm in\phi}$ and periodicity requires n is an integer. Sturm-Liouville guarantees completeness, which means any function $f(\phi)$ defined on $[-\pi, \pi]$ with at most finitely many discontinuities and no singularities can be written

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

in the sense that the norm of $\left(f - \sum_{-N}^{N} a_n e^{inx}\right) \underset{N\to\infty}{\longrightarrow} 0$, where the norm $|g|^2 = \displaystyle\int_{-\pi}^{\pi} |g(x)|^2 dx$.

The function $\bar{f} = \sum a_n e^{inx}$ generated from f is, of course, periodic in x with period 2π, even if f isn't. The usefulness of the expression depends, however, on f being periodic, or at least that the properties of f one is interested in are not affected by making it periodic. A prime example is in discussing the tone of a musical note. The note is approximately periodic with period 2π in the variable ωt. The fourier series is a description of the waveform in terms of the harmonics.

The fourier coefficients a_n corresponding to a function f are found by

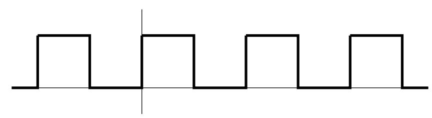
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\, dx = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n \int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = a_m.$$

If f is real, $a_m = a^*_{-m}$,
if f is symmetric, $a_m = a_{-m}$,
if f is antisymmetric, $a_m = -a_{-m}$.

Some examples:

Square Wave:

Consider $\quad f(x) = \begin{cases} V, x \in (0,\pi) \\ 0, x \in (-\pi, 0) \end{cases}$



as for a clock in some electronic designs.

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{V}{2\pi}\int_0^{\pi} e^{-inx}\, dx \\ &= \frac{V}{2\pi}\frac{e^{-inx}\big|_0^{\pi}}{-in} = \begin{cases} \quad -i\dfrac{V}{\pi m} & \quad m \text{ odd} \\ \quad 0 & m \text{ even} \neq 0 \end{cases} \\ &= \frac{V}{2} \qquad \text{for } m = 0 \end{aligned}$$

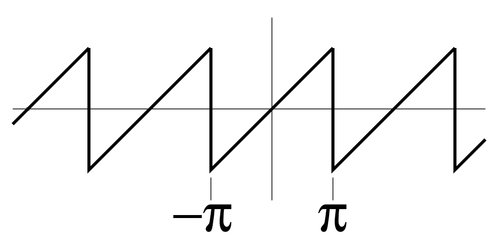
$$\text{so} \quad \bar{f}(x) = \frac{V}{2} + \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left[(2n+1)x\right]}{2n+1}.$$

A square wave (of 50% duty cycle) has only odd harmonics.

In any real electronic device $f(x)$ cannot really make an infinitely sharp transition at $x = 0$. Note $\bar{f}(0) = V/2$, the average $\dfrac{1}{2}\left(\lim_{x\searrow 0} + \lim_{x\nearrow 0}\right)$. This is a general feature of the behavior of a fourier series at a discontinuity.

Sawtooth:

The horizontal motion in a CRT TV set looks roughly like a sawtooth, $f(x) = x$ on $-\pi < x < \pi$, but periodic with period 2π (in ωt).



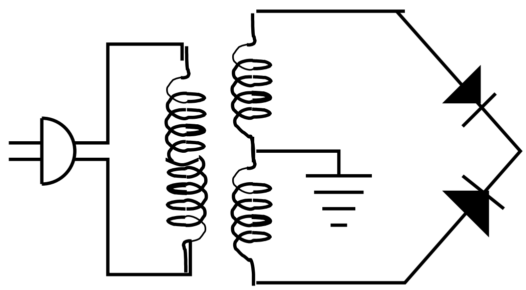
$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, e^{-inx}\, dx = \frac{i}{2\pi n}\int_{-\pi}^{\pi} x\,\frac{d}{dx}\, e^{-inx}\, dx \\ &= \frac{i}{2\pi n}\underbrace{x e^{-inx}\big|_{-\pi}^{\pi}}_{2\pi(-1)^n} - \frac{i}{2\pi n}\underbrace{\int_{-\pi}^{\pi} e^{-inx}\, dx}_{0} = \frac{(-1)^n i}{n} \qquad n \neq 0, \end{aligned}$$

and $a_0 = 0$.

$$\bar{f}(x) = \sum_{n=1}^{\infty} i(-1)^n \left[\frac{e^{inx}}{n} + \frac{e^{-inx}}{-n}\right] = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

Here we see all the harmonics. In both cases we see $a_n = \mathcal{O}(n^{-1})$ at large n, which is typical of functions with discontinuities.

Full wave rectifier:



$$V(t) = V_0|\sin\omega t|$$

Let $x{=}\omega t$, and Fourier transform $|\sin\omega t|$:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi}|\sin x|\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\; dx = \frac{2}{\pi} \\ (n\neq 0)\quad a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin x|\,e^{inx}\,dx = \frac{1}{2\pi}\int_0^{\pi}\sin x\left(e^{inx}+e^{-inx}\right)dx \\ &= \frac{1}{4\pi i}\int_0^{\pi}\left(e^{ix}-e^{-ix}\right)\left(e^{inx}+e^{-inx}\right)dx \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} e^{irx}\,dx &= 0 \quad \text{for } r \text{ even} \\ &= \frac{2i}{r} \quad \text{for } r \text{ odd.} \end{aligned}$$

so $\quad a_n = \dfrac{1}{2\pi}\left(\dfrac{1}{n+1} - \dfrac{1}{n-1} + \dfrac{1}{1-n} - \dfrac{1}{-1-n}\right) = \dfrac{-2}{\pi(n^2-1)} \quad$ for r even

and $a_n = 0$ for n odd. So $V/V_0 = \left[\dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos 2nx}{4n^2-1}\right] V_0$.

Note here the coefficients fall as n^{-2}, and are absolutely summable. This is because the function is continuous. Its derivative, however, falls with one less power of n so is not absolutely convergent, indicating a discontinuity (at 0 and π).

1.1 Integration and Differentiation

$$\begin{aligned} \text{If} \qquad f(x) &= \sum_{n=-\infty}^{\infty} a_n e^{inx} \\ \text{then} \qquad f'(x) &= \sum_{n=-\infty}^{\infty} i\, n\, a_n e^{inx} \end{aligned}$$

so if the fourier series for f converges it is easy to differentiate the function, but convergence of the series is made worse by the extra factor of n. On the other hand

$$\int_0^x f(t)\, dt = a_0 x + \sum_{n\neq 0} \frac{a_n}{in}\left(e^{inx} - 1\right).$$

this is not a periodic function unless $a_0 = 0$, *i.e.* $\int f = 0$ over one period, because otherwise integrating over many periods builds up $\int f$. But except for $a_0 x$ the new function is given by a fourier series with

$$\begin{aligned} b_n &= \frac{a_n}{in}, \qquad n \neq 0 \\ b_0 &= \sum_{n\neq 0} i\frac{a_n}{n}. \end{aligned}$$

Note the integral may have an absolutely convergent expansion even if the function does not.

Consider again the square wave

$$f(x) = \frac{V}{2} + \frac{2V}{\pi}\sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)x\right]}{2n+1}$$

which might constitute an ideal clock pulse in a digital electronic circuit. Real devices, however, have an upper limit to their frequency response, so a device that should be putting out an ideal waveform f is likely to put out a truncated version,

$$\bar{f}(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

where $a_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$, so

$$\bar{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sum_{n=-N}^{N} e^{in(x-t)}\, dt$$

The sum can be done, as it is a geometric series,

$$\frac{e^{-iN(x-t)} - e^{i(N+1)(x-t)}}{1 - e^{i(x-t)}} = \frac{\sin\left[(N+\frac{1}{2})(x-t)\right]}{\sin\frac{1}{2}(x-t)}.$$

For our square wave $f = V$ for $x \in (0,\pi)$, $f = 0$ for $x \in (-\pi, 0)$,

$$\bar{f} = \frac{V}{2\pi}\int_0^\pi \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin\frac{1}{2}(x-t)}\,dt.$$

For N large, the integrand is very large for $x - t \sim \mathcal{O}(1/N)$, and rapidly oscillating elsewhere. Let $y = Nx$ and change variables to $u = Nt$, so

$$\bar{f} \approx \frac{V}{2\pi}\int_0^{N\pi} \frac{\sin(y-u)}{\sin[\frac{1}{2N}(y-u)]}\,\frac{du}{N},$$

and only the region $y - u \ll N$ is significant. Thus we may expand the sine in the denominator,[1] As the

$$\bar{f} \approx \frac{V}{\pi}\int_0^\infty \frac{\sin(y-u)}{y-u}\,du = \frac{V}{\pi}\int_{-y}^\infty \frac{\sin v}{v}\,v = -\frac{V}{\pi}\mathrm{si}(-y) = +V + \frac{V}{\pi}\mathrm{si}(y).$$

Note $\bar{f}(0) = -\dfrac{V}{\pi}\mathrm{si}(0) = V/2$, as we would expect because the truncation does not affect the antisymmetry (other than the constant piece $V/2$). Instead of rising infinitely quickly, however,

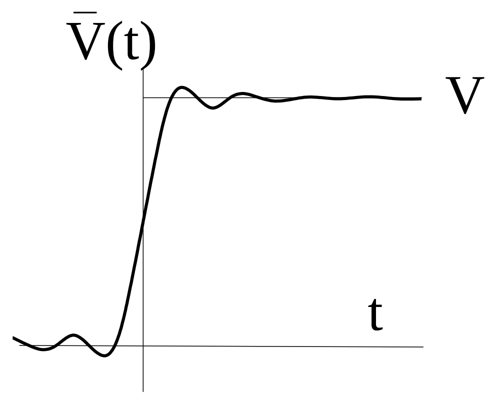
$$\bar{f}'(0) = \frac{d\bar{f}}{dx}(0) = N\frac{d\bar{f}}{dy}(0) = \frac{NV}{\pi}\frac{\sin(-y)}{-y} \to \frac{NV}{\pi},$$

[1] We need some facts about the si function, which we defined earlier as $si(x) = -\int_x^\infty \frac{\sin t}{t}\,dt$ so $si(0) = -\frac{1}{2}\ \mathrm{Im}\int_{-\infty}^\infty \frac{e^{it}-1}{t}\,dt$. As the integrand has no singularities, the integral along the real axis is minus the integral over a large circle in the upper half plane, $si(0) = +\dfrac{1}{2}\ \mathrm{Im}\displaystyle\int_0^\pi \frac{e^{iRe^{i\phi}}-1}{Re^{i\phi}}\,iRe^{i\phi}\,d\phi = -\frac{\pi}{2}$, as the exponential goes to zero for almost all ϕ and gives a finite contribution divided by R, so only the -1 contributes. Also, as the integrand is symmetric, $\int_{-\infty}^{-y} = \int_y^\infty$ and so $si(-y) = -\pi - si(y)$.

so for large N the voltage rises fast, $\propto N$. The maximum $\bar{f}$ takes, however, is when $\bar{f}' = 0 \Longrightarrow y = \pi$ (acutally $n\pi$, but $n = 1$ is the global maximum) with $\mathrm{si}(-\pi) = 0.281141$, so

$$V_{\max} = 1.08949V$$

independent of how high the cut-off frequency is. So adding more overtones fully, up to a larger limit n, does not help damp the overshoot. This is called the **Gibbs phenomenon**.



A technique for dealing with this overshoot problem is described in Arfken (2nd Ed.) under the name of Lanczos convergence factors, which we will not cover.