Physics 464/511 Lecture I Fall, 2016

1 Differential Equations

Now it is time to return to differential equations

Laplace's:	$\nabla^2\psi = 0$
Poisson's:	$\nabla^2\psi = -\rho/\epsilon_0$
Helmholtz:	$\nabla^2\psi + k^2\psi = 0$
Diffusion:	$\nabla^2\psi - \frac{1}{a^2}\frac{\partial\psi}{\partial t} = 0$
Wave Equation:	$\Box^2\psi = 0, \qquad \Box^2 := \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$
Klein-Gordon:	$\Box^2\psi - \mu^2\psi = 0$
Schrödinger:	$-\frac{\hbar^2}{2m}\left(\nabla^2 + V\right)\psi - i\hbar\frac{\partial\psi}{\partial t} = 0$

All are of the form $H\psi = F$, where H is a differential operator which could also depend on $\vec{r}$,

$$H\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}, x, y, z, t\right)$$

Simplifications:

1. All the equations are linear in ψ. This is crucial.

2. They are second order differential equations. In fact all of these are differential only through involving ∇^2 and $\partial/\partial t$. This is because the physics involved is rotationally invariant.

We will start with the results of the method of separation of variables. We already considered the Helmholtz equation in spherical coordinates r, θ, and ϕ. Recall that periodicity in ϕ required $\Phi(\phi) = e^{im\phi}$, with m an integer. This solution will appear whenever the equation and boundary conditions are symmetric under rotations about one axis.

In cylindrical coordinates $Z(z) \propto e^{az}$ for some real or imaginary a. There are no automatic boundary conditions here, but the particular problem may impose them, determining possible a's. For example, we might have a closed pipe with $Z(z) = 0$ at $z = 0$ and $z = L$, or we might require good behavior as $z \to \pm\infty$.

In spherical coordinates we found Legendre's equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta - \frac{m^2}{\sin^2\theta}\Theta = 0,$$

where I have replaced the constant Q by $\ell(\ell+1)$ for future convenience. Let $z = \cos\theta$, so $\dfrac{d}{d\theta} = -\sin\theta\dfrac{d}{dz}$. Then if $y(z) := \Theta(\theta)$,

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{dy}{dz}\right] + \ell(\ell+1)y - \frac{m^2}{1-z^2}y = 0$$

is another form of Legendre's equation. We may also write it as

$$\left(1-z^2\right)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left(\ell(\ell+1) - \frac{m^2}{1-z^2}\right)y = 0.$$

The radial coordinate satisfied $R(r)$ of Bessel's equation

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left(k^2r^2 - \ell(\ell+1)\right)R = 0$$

and a similar Bessel's equation arose in cylindrical coordinates. We will also want to consider the Laguerre equation which arises in the radial wave functions of the hydrogen atom, and Hermite's from the wave function for a harmonic oscillator. Chebyshev's is important in curve fitting in numerical analysis.

Because all of these equations are second order linear equations, they can be written

$$y'' + P(x)y' + Q(x)y = F(x),$$

where there may be values of x for which P and Q are singular. (*e.g.* $r = 0$ or $z = \pm 1$.)

If either P or Q (or both) are singular at $x = x_0$, x_0 is called a *singular* point. If not, it is called an *ordinary* point.

If x_0 is a singular point, but $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at $x = x_0$, x_0 is a *regular* singular point. Otherwise it is called an *essential* singularity.

We say that $x = \infty$ is an ordinary point, a regular singular point, or an essential singularity if, after changing variables to $z = 1/x$, $z = 0$ is that kind of point.
We are nearly always interested in equations where all singularities except at infinity are regular.

Our equation is *linear* in y. If $F(x) = 0$ it is also *homogeneous*. The problem of finding all solutions with a given $F(x)$ can be reduced to

- Find one solution.
- Find all solutions of the homogeneous equation with F set to zero. Then the general solution is the sum of the one solution with F and an arbitrary solution to the homogeneous equation.

1.1 Frobenius' Method

We will first try to solve the homogeneous equations by power series expansion about x_0. x_0 may be ordinary or a regular singular point, but not an essential singularity. For convenience only we assume $x_0 = 0$.

Let us look for a solution of the form

$$\begin{aligned}
y(x) &= x^k \sum_{n=0}^{\infty} a_n\, x^n, \quad \text{with } a_0 \neq 0 \\
&= \sum_{n=0}^{\infty} a_n\, x^{n+k} \\
y'(x) &= \sum_{n=0}^{\infty} (n+k)\, a_n x^{n+k-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2}
\end{aligned}$$

Now $\quad xP(x) = \sum_{r=0}^{\infty} b_r x^r$ and $\quad x^2 Q(x) = \sum_{r=0}^{\infty} c_r x^r \quad$ as these are analytic at

0, so

$$\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} + \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(n+k)a_n b_r x^{n+r+k-2}$$
$$+\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} a_n c_r x^{n+r+k-2} = 0.$$

Collecting powers of x^{p+k-2}

$$a_p(p+k)(p+k-1) + \sum_{n=0}^{p} a_n\big(b_{p-n}(n+k) + c_{p-n}\big) = 0.$$

For $p = 0$ we get the *indicial equation*

$$a_0\big[k(k-1) + b_0 k + c_0\big] = 0.$$

But $a_0 \neq 0$, so we get a condition on k, which determines the power we start off from.

This quadratic equation for k always has roots, so in the generic case with two distinct roots we will get two solutions, one from each root. Higher a_p can be determined recursively from

$$\left[(p+k)(p+k-1) + b_0(p+k) + c_0\right] a_p = -\sum_{n=0}^{p-1} a_n \left(b_{p-n}(n+k) + c_{p-n}\right)$$

provided $p + k$ does not satisfy the indicial equation. Thus each root will always give solutions except for the smaller of two roots differing by an integer. We also only get one solution if the roots are equal.

Let's consider some examples:

1. An ordinary point has $b_0 = 0$, $c_0 = c_1 = 0$. The indicial equation $k(k-1) = 0$ has solutions $k = 0, 1$. The equation for a_1, $k(k+1)a_1 + a_0 b_1 k = 0$ does not determine a_1 when $k = 0$, but is consistent. Therefore there are solutions with arbitrary a_0 and a_1 ($k = 0$), giving the two required solutions. As $k = 0$, the solution is analytic at the ordinary point.

2. Suppose $P(x)$ is an odd function of x and $Q(x)$ is an even function. Then the Parity operator $\psi(x) \to \tilde{\psi}(x) := \psi(-x)$ applied to a solution

will give a solution. Either this is the second solution (but with the same initial power of x^k) or it is the same solution multiplied by ± 1. The first case can be reduced to the second anyway, so the solutions will be either even or odd functions of x. As there are nonzero b's and c's only for even indices, the equation for the even a_n's never involve the odd ones, and vice versa. a_1 must satisfy its own indicial equation.

3. Bessel's equation is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

so $b_0 = c_2 = 1$, $c_0 = -n^2$, all others zero. The indicial equation $k(k-1) + k - n^2 = 0$ gives $k^2 = n^2$, $k = \pm n$. Thus for integral or half-integral n (which is what arises physically), the two roots differ by an integer, and if we try to find the a's corresponding to $k = -n$, we may not be able to solve the equation for $p = 2n$. But we have derived at least one solution with $k = +n$.

Fuchs's Theorem: This method, called *Frobenius' Method*, of expanding about an ordinary or regular singular point, always gives at least one solution as a power series expansion which converges up to the nearest singularity of $P(x)$, $Q(x)$, or $F(x)$.

1.2 Wronskian

If we have a linear n'th order ordinary differential equation,

$$y^{(n)}(x) + \sum_{j=0}^{n-1} P_j(x) y^{(j)}(x) = 0, \qquad \text{(where } d^{(0)}y \text{ means } y\text{)} \tag{1}$$

we know there ought to be n linearly independent solutions. Given n solutions, we want to know if they are linearly independent, and given fewer, we would like help finding the rest. One useful tool is the Wronskian.

First we consider n ordinary vectors $\vec{V}_j$ in an n-dimensional vector space, expressed in some basis. If the matrix $M_{kj} = (V_j)_k$ has a nonzero determinant, there is no non-zero vector a_j for which $\sum_j M_{kj} a_j = 0 = \left(\sum_j a_j \vec{V}_j\right)_k$, and hence the vectors are linearly independent.

Now consider the vector space of solutions of Eq. (1), and n functions $y_k(x)$ which solve that equation. Define the Wronskian as the determinant of the matrix of j'th derivatives, $j = 0, \ldots, n-1$ of the n functions y,

$$W(x) := \det \frac{d^{\ell-1} y_k}{dx^{\ell-1}} = \sum_{i_1,\ldots,i_n} \epsilon_{i_1,\ldots,i_n} \prod_{j=0}^{n-1} \frac{d^j y_{i_{j+1}}}{dx^j}.$$

If the Wronskian vanishes, even at one point, it means that at that point there is a nonzero a_j such that $\sum_j a_j y_j(x)$ vanishes along with its first $n-1$ derivatives, but then, as it satisfies Eq. (1), its n'th derivative also vanishes, and it is identically zero, so the n y_j's are not linearly independent.

If we differentiate the Wronskian, we can show $dW/dx = -P_{n-1}(x)W(x)$, and thus we can solve for $W(x)$, Thus W satisfies a first order differential equation with solution

$$W(x) = W(a) \exp - \int_a^x P_{n-1}(x')dx'.$$

For a general proof see my supplemental note "Wronskian", but we are interested in second order equations. For $n = 2$, $W(x) = \begin{Vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{Vmatrix} = f_1(x)\, f_2'(x) - f_1'(x)\, f_2(x)$. Suppose y_1 and y_2 are solutions of $y'' + P(x)y' + Q(x)y = 0$.

$$\begin{aligned} W' &= \frac{d}{dx}\left(y_1 y_2' - y_1' y_2\right) = y_1 y_2'' - y_1'' y_2 \\ &= y_1\left(-P y_2' - Q y_2\right) - y_2\left(-P y_1' - Q y_1\right) \\ &= -P(x)\left(y_1 y_2' - y_1' y_2\right) = -P(x) W \end{aligned}$$

So, at least for $n = 2$, we have verified our solution for $W(x)$.

Suppose we have a known solution y_1, and we want a linearly independent y_2. Overall normalization doesn't matter to us, so we can assume $W(a) = 1$, solve for $W(x)$, and find a first order equation

$$W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx} \frac{y_2}{y_1},$$

$$\text{so} \qquad y_2(x) = y_1(x) \left[\int_b^x \frac{W(x')}{y_1^2(x')}\, dx' \right].$$

[For $n > 2$, if we know $n-1$ solutions, noting that we know the Wronskian, which depends on y_n and only its first $n-1$ derivatives, we have an $(n-1)$'th order differential equation for y_n, which might be easier. But it is for $n = 2$ that we will find this most useful.]

1.3 Solutions of Bessel's Equation

Bessel's equation is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0.$$

So $P(x) = 1/x$, $Q(x) = 1 - \frac{n^2}{x^2}$, so $b_0 = 1, c_0 = -n^2, c_2 = 1$, all others vanish. The indicial equation is $k^2 = n^2$, so Frobenius' method gives us a solution from $k = n$. Successive a's are given by

$$\left[(n+p)^2 - n^2\right] a_p = -\sum_{j=0}^{p-1} a_j \left[\delta_{p-j,0}(j+n) + \delta_{p,j+2} - n^2\delta_{p,j}\right] = -a_{p-2},$$

or

$$a_p = \frac{-1}{p(p+2n)}a_{p-2}, \qquad \text{so } a_p = 0 \text{ for odd } p,$$

$$a_{2m} = \frac{(-1)^m\,(2n)!!a_0}{(2m)!!(2m+2n)!!} = \frac{(-1)^m\, n!a_0}{2^{2m}\, m!\,(m+n)!}.$$

Let $a_0 = 1/(2^n n!)$ in order to get Bessel's function normalized by standard convention. Then

$$a_{2m} = \frac{(-1)^m}{2^{2m+n}\, m!\,(n+m)!}$$

$$\text{and} \qquad y_1(x) = J_n(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{2m+n}.$$

This is called the Bessel function of the first kind.

Now let us turn to finding the second solution. As $P = 1/x$, $W(x) = \exp - \int^x P(x')dx' = e^{-\ln x} = 1/x$.

$$y_2(x) = J_n(x)\int^x \frac{1}{x' J_n^2(x')}dx'.$$

Writing the integrand in a Laurent expansion, the most singular term at $x = 0$ comes from $J_n \sim \frac{1}{n!\,2^n}x^n$, so

$$y_2(x) \sim n!\,2^n x^n \int^x (x')^{-1-2n}dx' = \frac{2^{n-1}}{(n-1)!}\,x^{-n}\left(1 + \mathcal{O}(x)\right) \qquad n \neq 0.$$

At first sight it looks like this is just what we would get from Frobenius, using the root $k = -n$, and indeed for n not integral that is correct. But for integer n the expansion of $1/x' J_n^2(x')$ will in general have a term $a_{-1}x^{-1}$, which on integration gives a $\ln x$ rather than a constant, multiplying J_n.

For $n = 0$, $J_0 \sim 1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots$,

$$\begin{aligned} y_2 &= J_0(x) \int^x \frac{1}{t} \underbrace{\left(1 - \frac{t^2}{4} + \frac{t^4}{64} + \cdots\right)^{-2}}_{1 + \frac{t^2}{2} + \frac{5}{32}t^4 + \cdots} dt \\ &= J_0(x) \left(\ln x + \frac{x^2}{4} + \frac{5}{128}x^4 + \cdots\right). \end{aligned}$$

The irregular solutions to the Bessel equation are called Neumann functions, $N_\nu(x)$ or $Y_\nu(x)$. As the Bessel equation arises from the Helmholtz equation in cylindrical polar coordinates, circular wave guides will have modes with radial dependence given by $J_n(\alpha r)$, or, for coaxial ones, where the solution needn't be regular at the origin, by a superposition of J and N.

1.4 Sturm-Liouville, Self-Adjoint differential equations

Consider a differential operator $\mathcal{L}$ such as

$$\mathcal{L}u(x) = p_0(x)\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_2(x)u(x) \tag{2}$$

which is of the same type we considered before, but with p_0 not divided out.

We consider $\mathcal{L}$ as a map on the space of functions defined on an interval $[a, b]$ ($a = -\infty$ or $b = +\infty$ is okay), where (a, b) contains no singular points, and within which p_0', p_0'', p_1', and p_2 are continuous.

If we defined an inner product $(v, u) = \int_a^b v^*(x)u(x)dx$, then the hermitian conjugate of $\mathcal{L}$ would be defined by $(v, \bar{\mathcal{L}}u) = (u, \mathcal{L}v)^*$, or $\int_a^b v^*(x)\bar{\mathcal{L}}u(x)dx = \int_a^b u(x)\mathcal{L}v^*(x)dx$, where I have assumed the p's are real (for real x). So

$$\int_a^b v^*(x)\bar{\mathcal{L}}u(x)dx = \int_a^b up_0\frac{d^2}{dx^2}v^*dx + \int_a^b up_1\frac{d}{dx}v^*dx + \int_a^b up_2v^*dx.$$

Integrating by parts and throwing away the end point contributions,

$$\int_a^b v^*(x)\bar{\mathcal{L}}u(x)dx = \int_a^b v^*\left[\frac{d^2}{dx^2}(p_0u) - \frac{d}{dx}(p_1u) + p_2u\right],$$

so

$$\bar{\mathcal{L}}u = \frac{d^2}{dx^2}(p_0 u) - \frac{d}{dx}(p_1 u) + p_2 u \tag{3}$$

is a good definition of an adjoint operator. Take this as a motivation rather than a proof, as we have ignored the surface terms (*i.e.* end point contributions.)

$\mathcal{L}$ is a self-adjoint operator if $\mathcal{L} = \bar{\mathcal{L}}$. But expanding (3),

$$\bar{\mathcal{L}}u = \left(\frac{d^2}{dx^2}p_0\right)u + 2\left(\frac{d}{dx}p_0\right)\frac{du}{dx} + p_0\frac{d^2u}{dx^2} - \left(\frac{d}{dx}p_1\right)u - p_1\frac{du}{dx} + p_2 u,$$

so equality (for all u) with (2) requires

$$2\frac{dp_0}{dx} - p_1 = p_1, \qquad \frac{d^2p_0}{dx^2} - \frac{dp_1}{dx} + p_2 = p_2,$$

or $\dfrac{dp_0}{dx} = p_1$, from which both follow.

If $\mathcal{L}$ is self-adjoint, $\mathcal{L}u = \dfrac{d}{dx}\left[p_0(x)\dfrac{du}{dx}\right] + p_2(x)u(x)$. Then we call $p = p_0$ and $q = p_2$.

Clearly not every 2nd order differential operator is self-adjoint, but each equation can be made so by finding a suitable function to multiply it by. If $y'' + P(x)y' + Q(x) = 0$, then $\mathcal{L}y = p_0 y'' + p_1 y' + p_2 = 0$ with $p_1 = Pp_0$, $p_2 = Qp_0$ and self-adjointness of $\mathcal{L}$ requires only $\dfrac{dp_0}{dx} = Pp_0$, or $\ln p_0 = \int P(x)dx$.

In our treatment of separation of variables, we got relations of the form $\mathcal{L}u(x) + \lambda w(x)u(x)$, where λ was an unknown separation constant and $w(x)$ was a known function necessary in order to get things to separate. For example, for Helmholtz in spherical coordinates, for $R(r)$ we have $\mathcal{L} = r^2\frac{d^2}{dr^2} + 2r\frac{d}{dr} - \ell(\ell+1)$ and $w(r) = r^2$, with $\lambda = k^2$. Multiplying by a function to make $\mathcal{L}$ self-adjoint will only change the form of $w(x)$, so we may as well consider only self-adjoint $\mathcal{L}$. We will assume $w(x) > 0$ on (a, b) for reasons to emerge.

Let us start with a very simple example. Take the equation Φ satisfies in axially or spherically symmetric problems:

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0.$$

Here $p = 1$, $q = 0$, $w = 1$, $\lambda = m^2$, and $a = 0$, $b = 2\pi$. ($\phi \to x$, $\Phi \to y$ or u or v). There are always some kind of boundary conditions to be imposed. Here they are periodicity. If $u = \Phi$ is an acceptible solution, it is periodic $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$.

We know the solutions are $\propto \cos mx$ or $\sin mx$. It is the boundary condition which forces m to be an integer. We could also consider $e^{im\phi}$, but we would still find m quantized to be an integer.

Let us define $\phi_m(x) = \frac{1}{\sqrt{2\pi}} e^{imx}$. Then

$$\int_0^{2\pi} \phi_m^*(x)\phi_n(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}\,dx = \delta_{m,n}.$$

Thus the ϕ's are *orthonormal* basis vectors of an infinite dimensional vector space.

How does this generalize to the general case

$$(\mathcal{L} + \lambda w)\,y = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + q(x)y(x) + \lambda w(x)y(x) \quad ?$$

a) the boundary conditions:

If $\mathcal{L}$ is to be considered Hermitian, we must have $(v, \mathcal{L}u) = (u, \mathcal{L}v)^*$ (with p, q real), or

$$\int_a^b \left\{ v^*(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + v^*(x)q(x)u(x)\right\} dx$$
$$= \int_a^b \left(u\frac{d}{dx}\left(p(x)\frac{dv^*}{dx}\right) + u(x)q(x)v^*(x)\right) dx$$

The qv^*u terms cancel. Integrating the first expression by parts,

$$v^*(x)p(x)\frac{du}{dx}\bigg|_a^b - \int_a^b \frac{dv^*}{dx}p(x)\frac{du}{dx} = u(x)p(x)\frac{dv^*}{dx}\bigg|_a^b - \int_a^b \frac{du}{dx}p(x)\frac{dv^*}{dx}.$$

The $\int$ terms cancel, so $\mathcal{L}$ is hermitean on these two functions if

$$v^*(x)p(x)\frac{du}{dx}\bigg|_a^b = u(x)p(x)\frac{dv^*}{dx}\bigg|_a^b.$$

This is a rather nasty condition to impose, for each u must satisfy it with each allowed function v in the space on which we can consider $\mathcal{L}$ to be hermitean. Here are some examples of how it is imposed.

(a) The case of Φ: All functions in the space are periodic, so $u^*(b) = u^*(a)$ and $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, and the same for v. As $p = 1$ here, both sides vanish.

(b) The harmonic oscillator problem in quantum mechanics, which gives the Hermite polynomials. $p = e^{-x^2}$, $w = e^{-x^2}$, $a = -\infty$, $b = +\infty$. u and v are required to vanish at infinity.

(c) The Legendre equation (for general m, for $x \in [-1, 1]$)

$$\underbrace{\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx}}_{\frac{d}{dx}\left(1-x^2\right)\frac{dy}{dx}} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0.$$

so $p = 1 - x^2$

The limits $\theta = 0, \pi$ correspond for $x = \cos\theta$ to -1 and $+1$, and y is required to remain finite there, with finite derivative, even though ± 1 are regular singular points. As $p \to 0$ at a and b, finiteness of du/dx is all that is needed.

(d) Bessel's equation $y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0$. Multiply by x, $\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(x - \frac{n^2}{x}\right) y = 0$, so $p = x$. Zero and ∞ are the singular points, so they are the natural a and b.

The boundary conditions are finiteness at the origin (where $p \to 0$) and vanishing sufficiently fast at infinity. (to be investigated later).

(e) Dirichlet or Neumann boundary conditions. If we require all solutions to vanish at the boundaries (Dirichlet) or to have their derivatives vanish at the boundaries (Neumann), then both sides vanish.

We have been discussing u as vectors and $\mathcal{L}$ as an operator, a kind of infinite matrix by analogy. The equation $\mathcal{L}u(x) = -\lambda w(x)u(x)$ is said in the book to have eigenfunctions u and eigenvalues λ. But an eigenvalue equation $Mv = \lambda v$ does not have a $-w(x)$ in it.

More accurately, we can say that u is an eigenvector of the differential operator $\mathcal{O} = -\frac{1}{w(x)}\mathcal{L}$. Then our equation is $\mathcal{O}u = \lambda u$, so u is really an eigenvector with eigenvalue λ. But $\mathcal{O}$ is not hermitian according to the definition of the inner product we used earlier, $(v,u) = \int v^*(x)u(x)\,dx$. We may instead define a different measure of distance on our space of functions,

$$\langle v,u\rangle := \int_a^b w(x)v^*(x)u(x)\,dx.$$

The w measure just cancels the $1/w$ in $\mathcal{O}$, so with respect to this new metric, $\mathcal{O}$ is hermitian. A measure needs to be positive, which is why we imposed that constraint of $w(x)$.

Now to remind you of some linear algebra[1]

(a) Any hermitian matrix $\mathcal{O}$ can be written $\mathcal{O} = UDU^{-1}$ with a diagonal $D_{ij} = \lambda_i\delta_{ij}$ and U a unitary matrix.

(b) The set of vectors $\sum_i e_i U_{ij} = \phi_j$, for each j, is a complete set of eigenvectors (with eigenvalues λ_j). That is, any vector $V = \sum a_j\phi_j$.

(c) The λ's are *real*, and the ϕ's corresponding to different λ's are orthogonal. (In fact, all the ϕ_j's I have defined are orthonormal.)

Now let us see how this applies to solutions

$$\mathcal{L}u_i + \lambda_i w\, u_i = 0.$$

Let u_j be another solution with λ_j. Then $u_j^*\mathcal{L}u_i + \lambda_i w u_j^* u_i = 0$. This is also true with $i \leftrightarrow j$, and the complex conjugate taken. $u_i\mathcal{L}u_j^* + \lambda_j^* w u_i u_j^* = 0$.

But we showed from the hermiticity of $\mathcal{L}$ that

$$(u_j, \mathcal{L}u_i) = \int u_j^*\mathcal{L}u_i = (u_i, \mathcal{L}u_j)^* = \int u_i\mathcal{L}u_j^*$$

so

$$\left(\lambda_i - \lambda_j^*\right)\int u_j^*(x)u_i(x)w(x)ds = 0.$$

If $i = j$ the integrand $|u_i^2|w(x)$ is positive so the integral cannot vanish, and $\lambda_i = \lambda_i^*$, or λ_i is real. If $\lambda_i \neq \lambda_j$, the integral must vanish, and $\langle u_j, u_i\rangle = 0$, the u's are orthogonal with measure w. If $\lambda_i = \lambda_j$ but u_i and u_j are linearly independent, they are not necessarily orthogonal, but span a two

[1] Of course you probably only saw these facts for finite-dimensional algebras. Hopefully they will apply to our infinite dimensional ones as well. We discuss completeness below.

dimensional space with some basis which can be chosen orthonormal with respect to $w(x)$.

What about completeness? Can any function $F(x)$ be written as $\sum_{n=0}^{\infty} a_n\phi_n$, where ϕ_n is a set of eigenfunctions? We say that ϕ_n is complete if there is a sequence a_n such that

$$\lim_{m\to\infty}\int_a^b \left|F(x)-\sum_{n=0}^{m} a_n\phi_n(x)\right|^2 w(x)=0.$$

We take ϕ_n to be an orthonormal set (even if there are degenerate eigenvalues). Then if $F=\sum_{n=0}^{\infty} a_n\phi_n$, $a_n=\int F(x)\phi_n^*(x)w(x)\,dx$. If we define the a_n's this way, even if we don't assume F is given by $\sum_{n=0}^{\infty} a_n\phi_n$, then

$$\begin{aligned}0 \;\le\; \int_a^b \left|F(x)-\sum_{n=0}^{\infty} a_n\phi_n\right|^2 w(x)\,dx \quad &= \int_a^b |F(x)|^2\, w(x)\,dx \\ -\sum_{n=0}^{\infty} a_n \underbrace{\int F^*\phi_n w}_{a_n^*} - \sum_{n=0}^{\infty} a_n^* \underbrace{\int \phi_n^* F w}_{a_n} &+ \sum_{m,n=0}^{\infty} a_n a_m^* \underbrace{\int \phi_n^*\phi_m w}_{\delta_{m,n}} \\ = \int_a^b |F(x)|^2\, w(x)dx - \sum_n a_n^* a_n \ge 0 &\qquad \text{Bessel's inequality}\end{aligned}$$

If the series $\sum |a_n|^2$, which is monotone increasing, reaches $\int F^2 w$, which bounds it, then $F=\sum a_n\phi_n$.
Theorem without proof: ϕ_n is complete. And can be chosen real.

The δ function:

Let $K(x,t)=\sum_m \phi_m(x)\phi_m^*(t)$. Then if $F(t)=\sum a_n\phi_n(t)$,

$$\int w(t)K(x,t)F(t)dt = \sum_{mn} a_n\phi_m(x) \underbrace{\int \phi_m^*(t)\phi_n(t)w(t)dt}_{\delta_{m,n}} = \sum a_n\phi_n(x) = F(x),$$

so $w(t)K(x,t)$ is the Dirac delta function $\delta(x-t)$

Finally a Green's function, and the solution of the inhomogeneous equation

$$\mathcal{L}\psi(x)+\lambda w(x)\psi(x) = -\rho(x).$$

Let $G(x,y) = \sum_n \frac{\phi_n(x)\phi_n^*(y)}{\lambda - \lambda_n}$. Applying $\mathcal{L}_x + \lambda w(x)$ to $G(x,y)$ (y is fixed)

$$\begin{aligned}
(\mathcal{L}_x + \lambda w(x))\, G(x,y) &= \sum_n \frac{\mathcal{L}_x \phi_n(x) + \lambda w(x)\phi_n(x)}{\lambda_n - \lambda} \phi_n^*(y) \\
&= \sum_n \frac{-\lambda_n w(x)\phi_n(x) + \lambda w(x)\phi_n(x)}{\lambda_n - \lambda} \phi_n^*(y) \\
&= -w(x) \sum_n \phi_n(x)\phi_n^*(y) = -\delta(x-y).
\end{aligned}$$

Thus $V(x) = \int G(x,y)\rho(y)\, dy$ satisfies

$$\mathcal{L}V(x) + \lambda w(x) V(x) = -\rho(x)$$

and we have found the inhomogeneous solution V.