Physics 464/511 Lecture E Fall 2016

Last time we introduced manifolds as a way to describe spaces that might be curved rather than Euclidean, emphasizing making statements about them which are true regardless of how one parameterizes the space. The abstract description of this method involves manifolds, atlases, tangent and cotangent bundles, and seems very abstract, but it mostly boils down to recognizing that covariant objects transform in definite ways when one changes from one coordinate description to another, and that the physical properties are described by various tensors[1]. Among these tensors, one particularly useful set are the n-forms. Today we will learn how to integrate them.

1 Integration

We introduced the notion of a 1-form as the change in a scalar function under an infinitesimal displacement. So it is immediately obvious that if we integrate the 1-form df along a path $C = \mathcal{P}(\lambda)$ parameterized by $\lambda \in [a, b]$, we have

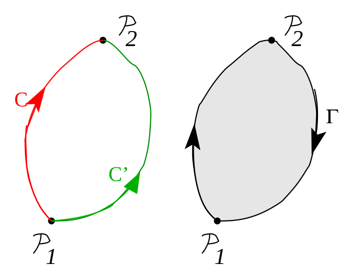
$$f(\mathcal{P}_2) - f(\mathcal{P}_1) = \int_C df = \int_C \frac{\partial f}{\partial q^j}\, dq^j = \int_a^b \frac{\partial f}{\partial q^j}\frac{dq^j}{d\lambda}\, d\lambda.$$

This is basically the fundamental theorem of calculus extended from the real line to arbitrary smooth curves. But this definition is a natural for the integration of a general 1-form along a path in a chart-independent way. If $\boldsymbol{\omega} = \omega_j\, dq^j$,

$$\int_C \boldsymbol{\omega} := \int_a^b \omega_j(\mathbf{q})\frac{dq^j}{d\lambda}\, d\lambda,$$

and this is independent of chart, and also of parameterization of the path.

But notice that if we integrate a general $\boldsymbol{\omega}$ along a different path C' from $\mathcal{P}_1$ to $\mathcal{P}_2$, we are not guaranteed to get the same answer. In fact, $\int_C \boldsymbol{\omega} - \int_{C'} \boldsymbol{\omega} = \oint_\Gamma \boldsymbol{\omega}$, the integral around the closed path. In cartesian coordinates, if we associate $\boldsymbol{\omega}$ with $\vec{V} = \omega_j \hat{e}^j$,



$$\int_\Gamma \boldsymbol{\omega} = \int_\Gamma \omega_j dx^j = \oint \vec{V}\cdot d\vec{\ell} = \int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{\sigma},$$

[1] In quantum mechanics, we also have spinors. We probably won't get to that.

where we used Stokes' theorem to get the last expression. Here $d\vec{\sigma}$ is an element of area of the surface S bounded by the closed path Γ.

What is $d\vec{\sigma}$? If we consider the surface embedded in Euclidean space to be described by parameters u and v, integrating over the surface $\int dudv$ is summing over little parallelograms with sides $d\vec{x} = \dfrac{\partial \vec{x}}{\partial u}\, du$ and $d\vec{x}\,' = \dfrac{\partial \vec{x}}{\partial v}\, dv$, with area and normal given by

$$d\vec{\sigma} = d\vec{x} \times d\vec{x}\,' = \frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\epsilon_{jk\ell}\hat{\mathbf{e}}_\ell.$$

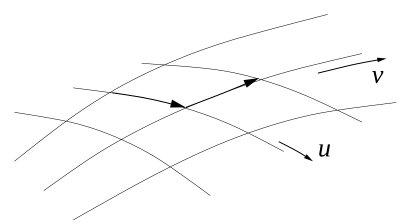
$$\begin{aligned}\text{Thus}\quad \int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{\sigma} &= \left(\epsilon_{\ell mn}\frac{\partial V^n}{\partial x^m}\right)\left(\epsilon_{\ell jk}\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}dudv\right)\\ &= \int_S \left(\frac{\partial V^k}{\partial x^j} - \frac{\partial V^j}{\partial x^k}\right)\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}dudv\end{aligned}$$

Notice that the 2-form $\boldsymbol{\omega}^{(2)}$ associated with the vector $\vec{\nabla}\times\vec{V}$ is

$$\frac{1}{2}\epsilon_{ijk}(\vec{\nabla}\times\vec{V})_i\; dx^j \times dx^k = \frac{1}{2}\epsilon_{ijk}\epsilon_{imn}\frac{\partial V^n}{\partial x^m}dx^j \times dx^k.$$

So we define in general the integral of a two form $\boldsymbol{\omega}^{(2)} = \frac{1}{2}B_{jk}dx^j \wedge dx^k$ over a surface to be

$$\int_S \boldsymbol{\omega}^{(2)} = \int B_{jk}(\mathbf{x}(u,v))\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}dudv.$$



We see that the integral $\int_S \boldsymbol{\omega}^{(2)} = \int_\Gamma \boldsymbol{\omega}$, where $\boldsymbol{\omega}^{(2)} = d\boldsymbol{\omega}$.

Finally, consider again an arbitrary vector field in cartesian coordinates in Euclidean space, but this time associate it with a 2-form

$$\vec{V} = v^i\mathbf{e}_i \Leftrightarrow \boldsymbol{\omega}^{(2)} = \frac{1}{2}B_{jk}\, dx^j \wedge dx^k, \quad \text{with } B_{jk} = \epsilon_{ijk}v^i$$

Let us integrate this over a closed two-dimensional surface S. Gauss assures us that $\int_S \vec{V}\cdot d\vec{\sigma} = \int_V \vec{\nabla}\cdot\vec{V}$. Now we saw last time that if $\vec{V} \Leftrightarrow \boldsymbol{\omega}^{(2)}$, $d\boldsymbol{\omega}^{(2)} = (\vec{\nabla}\cdot\vec{V})\boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = \frac{1}{6}\epsilon_{\ell jk}dx^\ell \wedge dx^j \wedge dx^k$. So we see that we should in general define the integral of a 3-form $\boldsymbol{\omega}^{(3)} = \frac{1}{6}B_{ijk}\, dx^i \wedge dx^j \wedge dx^k$ over a volume by $\displaystyle\int_V \boldsymbol{\omega}^{(3)} = \int B_{ijk}\left(\mathbf{x}(u,v,w)\right) \frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\frac{\partial x^k}{\partial w}\, dudvdw$. And we see that Gauss' law is now

$$\int_S \boldsymbol{\omega}^{(2)} = \int_V d\boldsymbol{\omega}^{(2)}.$$

Note that in this formula S is the boundary of the set V, and in Stokes' law $\int_S d\boldsymbol{\omega} = \int_\Gamma \boldsymbol{\omega}$, Γ is the boundary of S, and even in $f(\mathcal{P}_2) - f(\mathcal{P}_1) = \int_C df$ the left hand side can be viewed as the "integral" over the points $\mathcal{P}_2$ and $\mathcal{P}_1$, which is the boundary of the curve C, of the 0-form f, with suitable definitions of orientation. This sign issue is not new, because in even in Stokes' law we need to define the direction of $\int_\Gamma$ consistently with that of S. All of what we have shown is incorporated into the more sophisticated version of *Stokes' theorem*: If $\mathbf{R}$ is a closed p-dimensional region of a manifold $\mathcal{M}$ with boundary $\partial\mathbf{R}$ and if $\boldsymbol{\omega}$ is a smooth $(p-1)$-form defined on $\mathbf{R}$

$$\int_{\mathbf{R}} d\boldsymbol{\omega} = \int_{\partial\mathbf{R}} \boldsymbol{\omega}.$$

We have derived these versions of Stokes' theorem for Euclidean space in cartesian coordinates, but the p-forms are chart-independent, and the spaces over which they are integrated are subspaces of $\mathcal{M}$, so these hold regardless of the coordinates used. That they also hold for general differentiable manifolds regardless of curvature is less apparent, but is nonetheless true. But I do need to point out that the definition of the 3-form $\boldsymbol{\Omega}$ is not coordinate independent, and needs to be fixed. Let $\boldsymbol{\Omega}$ be the (hyper)-volume n-form on an n-dimensional Euclidean manifold given in cartesian coordinates by $\boldsymbol{\Omega} = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$. Because of the antisymmetry of the wedge product, this is also $\boldsymbol{\Omega} = \frac{1}{n!}\sum_{P\in S_n} \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_n}$, where the n index ϵ is defined by $\epsilon_{\mu_1,\mu_2,\ldots,\mu_n} = (-1)^P$, where P is the permutation that takes $(1,2,\ldots,n)$ into $\mu_1,\mu_2,\ldots,\mu_n$ and $(-1)^P$ means the sign of P, that is, -1 if P consists of an odd number of transpositions, $+1$ if an even number. But to be coordinate independent[2], for $\boldsymbol{\Omega} = \frac{1}{n!}\varepsilon_{\mu_1,\mu_2,\ldots,\mu_n} dq^{\mu_1} \wedge dq^{\mu_2} \wedge \ldots \wedge dq^{\mu_n}$,

[2]Note the distinction between ε and ϵ. There is no standard here — Wikipedia defines the Levi-Civita symbol to be the numerical one with values ± 1 and 0, and calls it ε, as does Vaughn, but calls the covariant ones E for what I call ε. Misner, Thorne and Wheeler use ε as I do, but $[\mu\nu\rho\sigma]$ for the numerical one. Weinberg's "Gravitation and Cosmology" has $\varepsilon^{0123} = 1$ with $\varepsilon_{0123} = -g$, thus covariant but differently normalized.

we need ε to transform covariantly in all its indices, which means[3]

$$\varepsilon_{\mu_1,\mu_2,\ldots,\mu_n} = \epsilon_{\nu_1,\nu_2,\ldots,\nu_n} \prod_{j=1}^{n} \frac{\partial x^{\nu_j}}{\partial q^{\mu_j}} = \epsilon_{\mu_1,\mu_2,\ldots,\mu_n} \ \det \frac{\partial x^j}{\partial q^k}.$$

But recall that in general coordinates q^j the metric tensor is $g_{jk} = \dfrac{\partial x^\ell}{\partial q^j}\dfrac{\partial x^\ell}{\partial q^k}$ so $g := \det g_{\cdot\cdot} = \left(\det \dfrac{\partial x}{\partial q}\right)^2$, and we have[4]

$$\varepsilon_{\mu_1,\mu_2,\ldots,\mu_n} = \sqrt{g}\ \epsilon_{\mu_1,\mu_2,\ldots,\mu_n}.$$

For pseudo-Riemannian spaces the metric may be negative, so then we use $\sqrt{-g}$ instead, with $\varepsilon_{\mu_1,\mu_2,\ldots,\mu_n} = \epsilon_{\mu_1,\mu_2,\ldots,\mu_n}$ for the Minkowski coordinates.

2 The Hodge Dual

We have seen that in three dimensional Euclidean space we have a natural association with a vector field $\vec{V}(\mathbf{x}) = v^i(\mathbf{x})\hat{e}_i$ and a 1-form $\boldsymbol{\omega}(\mathbf{x}) = \omega_i(\mathbf{x})dx^i$, with $v^i = \omega_i$, and also with a 2-form $\boldsymbol{\omega}^{(2)}(\mathbf{x}) = \frac{1}{2}B_{jk}(\mathbf{x})\ dx^j \wedge dx^k$, with $B_{jk} = \epsilon_{jki}\, v^i$. This gives us a kind of duality between 1-forms and 2-forms. More generally, in cartesian coordinates for n-dimensional Euclidean space, we define the Hodge * acting on a p-form to be an $n-p$ form

$$* \left(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}\right) = \frac{1}{(n-p)!}\epsilon_{i_1,i_2,\ldots,i_p,i_{p+1},\ldots,i_n} dx^{i_{p+1}} \wedge dx^{i_{p+2}} \wedge \cdots \wedge dx^{i_n}.$$

In non-cartesian coordinates we need to distinguish co- and contra-variant indices, which means we need to use the covariant ε rather than ϵ, and we need to raise p of the indices, so the covariant coefficients of the two forms are related by

$$* \left(\boldsymbol{\omega}^{(p)}\right)_{i_1,i_2,\ldots i_{n-p}} = \frac{1}{(n-p)!}\varepsilon^{j_1,j_2,\ldots,j_p}{}_{i_1,i_2,\ldots,i_{n-p}} \left(\boldsymbol{\omega}^{(n)}\right)_{j_1,j_2,\ldots j_p}. \tag{1}$$

[3]A discussion of ϵ with n indices and its use in defining determinants is given in `epsndeuc.pdf` and `determinant.pdf`, available on the supplementary notes page.

[4]Vaughn gives the name $\rho(x)$ to $\sqrt{g}$, as it is the density of space, if you like, as the volume integral is $\int \rho(q)dq^1 \ldots dq^n$.

The dual of the dual is almost the original p-form, except there is a sign,

$$**\boldsymbol{\omega}^{(p)} = (-1)^{p(n-p)} s\, \boldsymbol{\omega}^{(p)},$$

where s is the *signature* of the metric tensor, $+1$ for Riemannian space but -1 for Minkowski space. Thus the two forms associated with $\vec{V}$ are dual to each other in cartesian coordinates.

If we go to a general coordinate system instead of a cartesian one, coefficients of 1-forms and 2-forms transform differently. Note, in fact, that we had awkwardness in our vector $\leftrightarrow$ 1-form identification $v^i = \omega_i$, relating a covariant index to a contravariant one. But the Hodge dual permits us to give the relation covariantly.

3 The Laplacian

Many of the fields we consider in physics have partial differential equations involving the Laplacian. Acting on a scalar field, $\nabla^2\lambda = \vec{\nabla}\cdot(\vec{\nabla}\lambda)$. We have seen that the gradient $\vec{\nabla}\lambda$ is associated with the 1-form $\boldsymbol{\omega} = \dfrac{\partial\lambda}{\partial q^j} dq^j$, which is also associated with its 2-form dual, $\boldsymbol{\omega}^{(2)} = \dfrac{1}{2}\,\varepsilon^j{}_{k\ell}\,\dfrac{\partial\lambda}{\partial q^j}\, dq^k\wedge dq^\ell = \dfrac{1}{2} g^{jm}\varepsilon_{mk\ell}\,\dfrac{\partial\lambda}{\partial q^j}\, dq^k\wedge dq^\ell = \dfrac{1}{2} g^{jm}\sqrt{g}\;\epsilon_{mk\ell}\dfrac{\partial\lambda}{\partial q^j}\, dq^k\wedge dq^\ell$. But we know that the exterior derivative of a 2-form is associated with the divergence, so

$$\begin{aligned} d\boldsymbol{\omega}^{(2)} &= \frac{1}{2}\,\epsilon_{mk\ell}\,\frac{\partial}{\partial q^i}\left(g^{jm}\sqrt{g}\,\frac{\partial\lambda}{\partial q^j}\right) dq^i\wedge dq^k\wedge dq^\ell \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^i}\left(\sqrt{g}\,g^{ij}\frac{\partial\lambda}{\partial q^j}\right)\boldsymbol{\Omega}, \end{aligned}$$

where $\boldsymbol{\Omega} = \sqrt{g}\,dq^1\wedge dq^2\wedge dq^3$, the volume 3-form we considered earlier. Thus we have

$$\nabla^2\lambda = \vec{\nabla}\cdot\vec{\nabla}\lambda = \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^i}\left(\sqrt{g}\,g^{ij}\frac{\partial\lambda}{\partial q^j}\right).$$

4 Some Things Postponed

We have developed the fundamentals of a Riemannian manifold and of differential forms, laying the groundwork that we would want to go on to discuss

General Relativity. The next step in that direction would be to discuss parallel transport and covariant derivatives, and then curvature. This is very interesting stuff, and useful in considering gauge field theories as well as general relativity, and maybe if we have time later, we will come back to this.

If you are impatient, there are lots of books on general relativity that you can persue on your own, though it is not easy going. I learned it mostly from Misner Thorne and Wheeler, with Weinberg's "Gravitation and Cosmology" for some topics. I tried teaching it from Wald, but that was rough going. A book by Carroll is available free on the web `arxiv.org/pdf/gr-qc/9712019` and has a good bibliography of other books on pages v-vi. We have a graduate course, 617: General Theory of Relativity, scheduled to be taught next spring (2017). But for now, we will turn to more mundane uses of these concepts.

5 Orthogonal Coordinates

We are accustomed to giving our vector components using orthonormal basis vectors, but we have recently been describing them on our manifold using 1-forms with basis elements dq^i, with a scalar product $\mathbf{g}(u, v) = g_{ij}u^iv^j$. In general these are neither orthogonal nor normalized. As g is a positive definite symmetric matrix, we can always find unit vectors[5] $\tilde{e}_i$ with $\mathbf{g}(\tilde{e}_i, \tilde{e}_j) = \delta_{ij}$, and dx^i dual to $\tilde{e}_i$. This is possible independently at every point, but with no guarantee that dx^i can be extended to a finite region as coordinates. In fact, that is only possible in a flat space, giving us cartesian coordinates, and even there, these may not be the most helpful choice.

What **are** helpful choices are *orthogonal coordinates.* In general, for each of the coordinates q^j and each point in the manifold $\mathcal{P}_0$, the equation $q^j(\mathcal{P}) = q^j(\mathcal{P}_0)$ determines a surface (hypersurface of dimension $n-1$) containing the point $\mathcal{P}_0$, at least in a neighborhood of $\mathcal{P}_0$. Suppose it is possible to describe our space, or at least a chart covering a part $\mathcal{U}$ of our space, with n coordinates q^j such that the hypersurfaces corresponding to the n coordinates are orthogonal. That means there are no off-diagonal elements,

[5]If we are in a pseudo- Riemannian (*e.g.* Minkowski) space rather than Riemannian, replace δ_{ij} with the flat-space metric $\eta_{\mu\nu}$.

so[6]

$$\mathbf{g} = \sum g_{jk}\, dq^j \otimes dq^k = \sum_{k=1}^{n} (h_k\, dq^k)^2$$

We have used the fact that the Riemannian metric is positive definite to set the diagonalized matrix elements to h_k^2. For a pseudo-Riemannian space the signature of $\mathbf{g}$ is fixed to that of η, so again h_j is real. Note that the $\sqrt{g}$ (or $\sqrt{-g}$) in the volume n-form is now $\prod_k h_k$.

[Note: at this point we will temporarily turn off the summation convention — no sums unless explicitly indicated.]

[Also, we are now going to consider three-dimensional Euclidean space. Thus $g_{ij} = h_i^2 \delta_{ij}$, $g^{ij} = h_i^{-2}\delta_{ij}$, and $\sqrt{g} = h_1 h_2 h_3$.]

Thus the 3 dq^j's determine orthogonal vectors at the point $\mathcal{P}$, and we can define unit vectors $\hat{e}^j$ corresponding to $h_j dq^j$. The gradient of a scalar f is

$$\vec{\nabla} f = \sum_j h_j^{-1} \frac{\partial f}{\partial q^j} \hat{e}^j.$$

Consider a vector $\vec{V} = \sum_j v_j \hat{e}^j$ which corresponds to the 1-form $\boldsymbol{\omega}_V^{(1)} = \sum_j v_j h_j dq^j$ and its dual 2-form $\boldsymbol{\omega}_V^{(2)} = *\boldsymbol{\omega}_V^{(1)} = \frac{1}{2}\sum_{jk\ell} \varepsilon^j{}_{k\ell} v_j h_j dq^k \wedge dq^\ell$. Note $\varepsilon^j{}_{k\ell} = g^{jj}\varepsilon_{jk\ell} = \dfrac{h_1 h_2 h_3}{h_j^2}\epsilon_{jk\ell}$. Now $\boldsymbol{\omega}_V^{(1)}$ and $\boldsymbol{\omega}_V^{(2)}$ have exterior derivatives

$$\begin{aligned}
d\boldsymbol{\omega}_V^{(1)} &= \sum_{jk} \frac{\partial v_j h_j}{\partial q^k}\, dq^k \wedge dq^j \quad \Leftrightarrow \quad \frac{1}{2}\sum_{jk\ell} \varepsilon^j{}_{k\ell} (\vec{\nabla}\times\vec{V})_j h_j dq^k \wedge dq^\ell \\
&\qquad\qquad = \frac{1}{2}\sum_{jk\ell} \frac{h_1 h_2 h_3}{h_j^2} \epsilon_{jk\ell} (\vec{\nabla}\times\vec{V})_j h_j\, dq^k \wedge dq^\ell \\
\text{and} \quad d\boldsymbol{\omega}_V^{(2)} &= \frac{1}{2}\sum_{ijk\ell} \frac{\partial}{\partial q^i}\left(\frac{h_1 h_2 h_3}{h_j^2}\, \epsilon_{jk\ell}\, v_j h_j\right) dq^i \wedge dq^k \wedge dq^\ell \\
&\quad \Leftrightarrow \quad \vec{\nabla}\cdot\vec{V}\ h_1 h_2 h_3\ \epsilon_{ik\ell}\ dq^i \wedge dq^k \wedge dq^\ell
\end{aligned}$$

$$\begin{aligned}
\text{Thus} \qquad \vec{\nabla}\times\vec{V} &= \sum_{ijk} \epsilon_{ijk} \frac{1}{h_i h_j} \frac{\partial (h_j v_j)}{\partial q^i}\ \hat{e}_k \\
\text{and} \qquad \vec{\nabla}\cdot\vec{V} &= \frac{1}{h_1 h_2 h_3} \sum_j \frac{\partial}{\partial q^j}\left(\frac{h_1 h_2 h_3}{h_j} v_j\right).
\end{aligned}$$

[6] For a Riemannian space. If pseudo-Riemannian, we need $\sum_{jk} \eta_{jk} h_j h_k dq^j dq^k$ instead.

Finally, the laplacian of a scalar is

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i} \left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial f}{\partial q^i} \right).$$

6 Forms in Special Relativity

Even if we deal simply with flat Minkowski space in cartesian coordinates, the language of n-forms is useful. Let us review four-dimensional language of special relativity, and Maxwell's equations in free space.

We describe space-time positions with a 4-vector (really not a vector) x^μ, $\mu = 0, 1, 2, 3$ with $x^0 = ct$, and define the invariant $(d\tau)^2 = (dx^0)^2 - \sum_{j=1}^3 (dx^j)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where the Minkowski metric tensor[7] $\eta_{00} = 1$, $\eta_{ij} = -\delta_{ij}$, $\eta_{0j} = \eta_{j0} = 0$. The three dimension variables are $\vec{x} = x^j$, but for the momentum $\vec{p} = p_j = -p^j$. Scalar products $W \cdot V = \sum_\mu W^\mu V_\mu = W^0 V^0 - \vec{W} \cdot \vec{V}$. The zeroth component of the momentum is $p^0 = p_0 = E/c$, where E is the energy. Thus for a free particle of mass m, $p^2 = \left(\frac{E}{c}\right)^2 - \vec{p}^{\,2} = m^2 c^2$, independent of the reference frame.

Maxwell's equations in free space (where $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{H} = \vec{B}/\mu_0$, and $\epsilon_0 \mu_0 = c^{-2}$)

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

In terms of $\vec{E}$ and $\vec{B}$ the Lorentz transformation properties are somewhat involved, but all becomes clear if we combine things into covariant objects. Define the *field strength tensor*

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

or better yet the 2-form $\boldsymbol{F} = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$. Also define a 4-vector $J^\mu = (\rho/\epsilon_0, \vec{J})$. Let's ask what the exterior derivative of $\boldsymbol{F}$ is:

$$d\boldsymbol{F} = \frac{1}{2} \frac{\partial F_{\mu\nu}}{\partial x^\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu.$$

[7] The choice of overall sign is immaterial, but ours is sometimes called the west-coast convention. Note that this is not a 2-form, but a symmetric tensor in $\mathcal{T}^* \otimes \mathcal{T}^*$.

The coefficient of $dx^1 \wedge dx^2 \wedge dx^3$ in this expression is $\frac{1}{2}\epsilon_{jk\ell}\frac{\partial F_{k\ell}}{\partial x^j}$, and $F_{k\ell} = -\epsilon_{k\ell m}B_m$, so this component is $-\frac{1}{2}\epsilon_{jk\ell}\epsilon_{k\ell m}\frac{\partial B_m}{\partial x^j} = -\vec{\nabla}\cdot\vec{B} = 0$. The coefficient of $dx^0 \wedge dx^j \wedge dx^k$ is $\frac{1}{c}\frac{\partial F_{jk}}{\partial t} + \frac{\partial F_{0j}}{\partial x^k} + \frac{\partial F_{k0}}{\partial x^j} = -\frac{1}{c}\epsilon_{jk\ell}\dot{B}_\ell + \frac{\partial E_j}{\partial x^k} - \frac{\partial E_k}{\partial x^j} = -\epsilon_{jk\ell}\left(\frac{1}{c}\dot{B}_\ell + (\vec{\nabla}\times\vec{E})_\ell\right) = 0$. So two of Maxwell's equations, the ones without sources, tell us that that $d\boldsymbol{F} = 0$, and therefore $\boldsymbol{F}$ is a closed 2-form. We have seen that any exact form is closed. But it is also true[8] that if a form is closed through a contractible region, it is exact. So as $\boldsymbol{F}$ is closed everywhere in space-time, it must be exact, that is, it must be the exterior derivative of a 1-form $\boldsymbol{A} = A_\mu dx^\mu$. If we identify $A_0 = \Phi$, the electrostatic potential, and $A^j = \vec{A}$, we have $d\boldsymbol{A} = \frac{\partial A_\mu}{\partial x^\nu}dx^\nu \wedge dx^\mu = \boldsymbol{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, so $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. In terms of three-dimensional notation, $E_k = F_{k0} = -\frac{1}{c}\dot{A}_k - \frac{\partial \Phi}{\partial x^k}$ or $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi$, as usual, and $B_k = -\frac{1}{2}\epsilon_{kij}F_{ij} = \vec{\nabla}\times\vec{A}$, again as it should be.

What about the other two Maxwell equations? Consider the Hodge dual of $\boldsymbol{F}$, $\mathcal{F} := *\boldsymbol{F} = \frac{1}{2}\mathcal{F}_{\rho\sigma}dx^\rho \wedge dx^\sigma$, with

$$\mathcal{F}_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Notice this is just what you get by interchanging $\vec{B}$ with $-\vec{E}$, so the exterior derivative components are $\vec{\nabla}\cdot\vec{E}$ and $\epsilon_{jk\ell}\left(\frac{1}{c}\frac{\partial E_\ell}{\partial t} + (\vec{\nabla}\times\vec{B})_\ell\right)$, or $4\pi\rho$ and $\frac{4\pi}{c}\vec{J}$, which is to say $d*\boldsymbol{F} = *\boldsymbol{J}$. So we have rephrased the tee shirt:

$$\boxed{\begin{aligned} d\boldsymbol{F} &= 0 \\ d*\boldsymbol{F} &= *\boldsymbol{J} \end{aligned}}$$

Much more elegant!

[8] Poincaré lemma. Contractible means that the form is defined (and closed) in a region which can be continuously contracted within the itself to a point, such as a ball but not a donut.