Physics 464/511   First Lecture   Sept. 7, 2016

Mathematical Physics

"God used beautiful mathematics in creating the world."

Paul Dirac

Physics is an empirical science, but it is also a playground of mathematical ideas of all levels of abstraction, from simple arithmetic to cohomologies. This course is designed to cover many of the mathematical ideas which are useful in physics. Most of the course will cover ideas useful in the dynamics of fields, ideas which are also covered in your math courses on partial differential equations, complex variables, linear algebra, and differential geometry, as well as in physics courses on Classical Mechanics, Electromagnetism, and Quantum Mechanics. Some would include group theory in a course like this, but we have a separate course (618) so I will leave most of that large topic for that course.

We physicists tend to be less formal and rigorous than our mathematics colleagues. Indeed, many physicists before or early in the 20th century were rather distainful of mathematics other than differential equations. But throughout that century and since, we have found that more and more abstract mathematics turns out to be the basis of physics.

Those of you who have told me your backgrounds seem to have had pretty good backgrounds in linear algebra, vector calculus, analysis such as series, and differential equations, but not much background in complex variables, special functions, fourier analysis, and the tools of differential geometry. So while the first set of topics is essential and underlies all that we will discuss, I will review it only very briefly and spottily, and spend more time on the second set of topics.

Nonetheless, we should begin by asking what kinds of mathematical entities we need to discuss, and make sure we are familiar with some formal definitions and mathematical/logical notation.

1 Mathematical Preliminaries

As children, we begin our mathematical careers with the positive integers $\mathbb{Z}^+$ or $\mathbb{N}^+$, sometimes called the *counting numbers*. This is our first step

in quantifying the world. We first learn to add them, with the important properties of

1) *closure*: Addition is a *closed binary* operation:[1] $+ : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, and $\forall x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, x + y \in \mathbb{Z}^+$.

2) Addition is *commutative*: $\forall x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, x + y = y + x$.

3) Addition is *associative*: $\forall x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, \forall z \in \mathbb{Z}^+, (x+y)+z = x+(y+z)$. What this means is that if you take the number you get by adding x and y, and adding z to this number, you get the same thing as if you first add y and z, and take the result and add it to x. This is not automatically true of all binary operations.

A little later we learn to multiply, and discover multiplication is also a closed binary commutative associative operation, written $x \cdot y$ or just $x\,y$. In addition, multiplication has an *identity*, an element in $\mathbb{Z}^+$ called 1, with $\forall x \in \mathbb{Z}^+, 1 \cdot x = x$.

The same cannot be said of the positive integers $\mathbb{Z}^+$ under addition, but if we add zero to the set, to make the *natural numbers* $\mathbb{N} = (0, 1, 2, \ldots)$, which are the non-negative integers, then 0 is the identity for addition, $\forall x, 0 + x = x = x + 0$.

These properties make $\mathbb{Z}^+$ and $\mathbb{N}$ *semigroups* under the multiplication operation, and $\mathbb{N}$ a semigroup under addition. A semigroup needs only an associative binary operation, not necessarily commutative.

One more property: multiplication is *distributive* over addition, that is,

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, \ x \cdot (y + z) = x \cdot y + x \cdot z.$$

So far, we do not have inverses. But if we extend our set by adding in the negative integers, to make the set of all integers, $\mathbb{Z}$, then each element x of $\mathbb{Z}$ has an inverse under addition, $-z$, in $\mathbb{Z}$, such that $-z + z = 0$. This makes $\mathbb{Z}$ an *Abelian group* under addition. The expansion preserves the semigroup

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[1] Some useful symbols, logical and set-theoretic: $f : A \to B$ means f is a function mapping elements in the set A to element in B, while $f : x \mapsto y$ means f maps the element x (of the set A) into the element y (of the set B). $\forall$ means "for all"; $x \in S$ means x is an element of the set S; $\exists$ means "there exists (a)"; { } is the set of what's inside the braces; $\ni$ means "such that", but so does | inside { } , *e.g.* $\mathbb{N}^+ = \{n \in \mathbb{Z} | n > 0\}$. Of course | also is needed for absolute value or magnitude, as well as "evaluated at" after a derivative. For intervals, $(a, b) = \{x | a < x < b\}$ and changing the (to [or) to] means include the end-point. But (a, b) could also mean an ordered pair. And of course (and) can be just nesting indicators in an equation, as in $3(a + b)$. And $\to$ can mean other things as well, but not "implies", for which I will use $\Rightarrow$.

property under multiplication, and these properties, including the distribution law, makes $\mathbb{Z}$ a *commutative ring* with identity. (commutative because multiplication on $\mathbb{Z}$ is commutative), and the identity is 1. More generally, a *ring* needs an abelian addition and just a closed associative distributive (on both sides) multiplication.

Can we do something to make the multiplication a group as well? That would require inverses for all elements. So we would need to throw in all the rational numbers. But even then, 0 does not have an inverse. Throwing in the rationals (fractions p/q with p and q integers, $q \neq 0$, and no common divisors of p and q) makes $\mathbb{Q}$, the set of rational numbers, which is a field.

A *field* (for mathematicians) is a commutative ring with identity for which a multiplicative inverse exists for every element other than 0.

These mathematical fields will be very important. The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are also fields[2] and indeed are the ones we will be most interested in using. Real numbers are not so easily described in terms of the natural numbers, though the positive real numbers are just how long a line can be. One can think of the extensions of $\mathbb{Z}^+ \to \mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$ to be due to requiring solutions to the equations $a + x = a$, $a + x = 0$, $a \cdot x = b$, where in each case a and b are members of the previous set. More elements can be found similarly from $\mathbb{Q}$, or even $\mathbb{Z}$, for example the solution to $x^2 = 2$, historically the first irrational (and top-secret) number, together with all other real roots of polynomials with integer coefficients, form the *algebraic numbers* which is a field, but not one for which I know any use in physics. But the reals is a bigger set than the algebraic numbers, and to define them formally we need to consider all limits of Cauchy sequences of rational numbers.

Many, though not all, of the quantities of physics are given by real number multiples of some measure with units, such as the mass of a particle as a positive real number times 1 kg. Some of these quantities are quantized, such as charge, so in the right units they are described by integers rather than more general real numbers, and note that some quantities can take either sign while some are inherently positive. But some quantities live in bigger spaces. For example, the electric field at a point has direction as well as magnitude, and lives in a three-dimensional *vector space*. Also, the electric field at a point is just a small part of the full electric field[3], a vector-

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[2] The integers modulo a prime p are also fields, known as fields of characteristic p. But I doubt we will need them.

[3] This is a physicist's, not a mathematician's, field. Sorry, two completely different

valued *function* from spacetime, $\vec{E}(\vec{r}, t)$. Some objects live in spaces which are not vector spaces — for example, the possible rotations of a rigid body are elements of a group (SO(3)), so are neither real variables nor vectors. Elements of a group can be multiplied (composition) but not added to give new elements in the group.

And there are quantum mechanical operators, which form an algebra, with addition and multiplication, but no inverses in general. We will define that better in homework 2.

So we are interested in quantities that live in different spaces, and are elements of sets of such quantities. Let's first check and review our notation. If A and B are sets

$x \in A$	the element x is a member of the set A
$A = B$	the set A contains exactly the same elements as the set B
$A \subset B$	all elements in A are also in B
$A \supset B$	same as $B \subset A$
$A \cup B$	union of A and B, all elements in either or both sets
$A \cap B$	intersection of A and B, all elements that are in both sets
$A - B$	set of those elements of A that are not in B
$A \times B$	the set of pairs of elements, one from set A and one from set B. $A \times B$ is called the *direct product* of A and B.

The null set $\emptyset$, also called the empty set, has no elements. The number of elements in a set is called its *cardinality*. It can be a finite nonnegative integer, but it can also be $\aleph_0$, the cardinality of the set of natural numbers, or $\mathfrak{c}$, the cardinality of the real numbers, and there are still larger cardinalities. Two sets have the same cardinality if there is a 1-to-1 correspondance between them. This has some strange consequences for infinite sets. For example, the cardinality of the positive even integers is the same ($\aleph_0$) as that of all integers, even though the latter would seem to have 4 times (plus 1) as many elements. In fact, the cardinality of the rationals is also to same — all of these have $\aleph_0$ members. But the cardinality of the reals, $\mathfrak{c}$, is greater.

It is not known whether there is a cardinality between $\aleph_0$ and $\mathbf{c}$. A set with finite cardinality or cardinality $\aleph_0$ is *countable*, though some people define countable as requiring it be infinite.

A subset A of a set U $(A \subset U)$ is a set that contains zero or more elements of U and nothing else. Note $\emptyset$ and U are two subsets of U, but if we

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meanings for the same word.

exclude them we have the *proper subsets* of U. In all, there are 2^n distinct subsets of a set on cardinality n.

As we have mentioned, some sets of elements can have some operations defined on them, such as addition, or other structures. Then we call the set a *space.* One form of such structure consists of operations which map elements, or pairs of elements, of the space back into the space, as we discussed for addition and multiplication of groups and semigroups and fields. The structure might also involve several distinct spaces, as for a vector space, where a vector can be multiplied by a scalar (*i.e.* , an element of a different space which is a field) to produce another vector.

Another form of structure is a property that is associated with subsets. The most important is the notion of a *topological space*, for which certain subsets are declared *open,* while others might not be. This notion is pretty abstract in general, but we will principally concerned with metric spaces in which there is a distance defined between any two elements. In a metric space the elements are often called points, and the distance is a nonnegative real number.

We will have more formal mathematics to lay out, but this is pretty dry, so let's postpone that and begin to discuss (physicists') fields and differential operators and equations for them.