Physics 464/511 Homework #9 part 2
Due: Nov. 23, 2016 at 1:40 P. M.

Project #2

Note: This project is to be worked on in two groups, as discussed.

Each group should meet soon to lay out the steps necessary to complete the project, and divide up the work. Unlike ordinary homework, where though communication is encouraged, each individual is expected to write up all parts himself, in a project it is acceptible for each member to have only read through and understood each part. Everyone is responsible for it being correct and clear.

The project is to be written up consistently, coherently, and neatly, preferably on a computer, even more preferably in TEX or LATEX. Each member is responsible for proofreading before submission.

The project considers hyperspherical coordinates in four dimensions, and the separation of the laplacian (or rather Helmholtz' equation) into radial and angular parts.

1 [25 pts] If we lived in a Euclidean space of dimension $N > 3$, we would still want to have something like spherical, or rather hyperspherical, coordinates, one r the distance from an arbitrary point $\vec{r}$ to the origin and $N-1$ angles θ_i to specify the direction. We will also use supplementary values ρ_i, with $\rho_1 = r$.

If we consider what we did in 3 dimensions, it is first to define θ_1 the angle between a fixed (z_1) axis and $\vec{r}$. For a fixed ρ_1, the subspace of $|\vec{r}| = \rho_1$ is a 2-sphere S^2, and the intersection of this with the plane $z_1 = \rho_1 \cos\theta_1$ is a circle S^1 of radius $\rho_2 := \rho_1 \sin\theta_1$. We naturally describe the points on that S^1 as always with an angle θ_2 with respect to another arbitrary (x) axis. Of course we usually rename θ_1 as θ and θ_2 as ϕ, unless we are mathematicians and do the reverse, and z_1 as z and z_2 as x. For completeness, call the last, left over dimension, z_n, though in 3-D we call this y.

But the general idea is to start with the overall radius, and then describe the coordinates on a sequence of spheres S^{D-j}, $j = 1..D-1$ which are the intersections of the the current sphere of radius ρ_j with a hyperplane with one previously unspecified coordinate given $\in [0, \rho_j]$ as $\rho_j \cos\theta_j$. This means each $\theta_j \in [0, \pi]$ except the last, $\theta_{D-1} \in [0, 2\pi)$ in order to get both of the remaining coordinates correct. If this seems abstract, work it through with the usual 3-dimensional space.

(a) [4 pts] Now work this through for four dimensional space, describing each point with $(r, \theta_1, \theta_2, \theta_3)$, and give the expressions for z_j in terms of $q^j := (r, \theta_1, \theta_2, \theta_3)$.

(b) [5 pts] Work out the metric tensor g_{jk} in these hyperspherical coordinates, $\sum_{j=1}^{4} (dz_j)^2 = \sum_{j,k=1}^{4} g_{jk}\, dq^j\, dq^k$.

(c) [4 pts] Find the laplacian[1] $\Box$ in these hyperspherical coordinates.

(d) [12 pts] Consider the Helmholtz equation $(\Box + k^2)\psi = 0$ by separation of variables, and find the equations for the dependence of each factor. Use the requirement that $\Theta_3(\theta_3)$ must be periodic with period 2π, and suggest what conditions might restrict the other separation constants.

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[1] $\Box$ is usually used with the Minkowsky metric, $\Box := \nabla - \frac{1}{c^2}\frac{d^2}{dt^2}$, but here we mean just the four-dimensional $\sum_{k=1}^{4} \frac{d^2}{dx_k^2}$.