Physics 464/511 Homework #6
Due: Oct. 31, 2016 at 5:00 P. M.

1 [8 pts] The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.
(a) [4 pts] Assuming that the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

(b) [4 pts] Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

and give a geometric interpretation.
[Hint: what are the normals to the curves $u =$ constant and $v =$ constant?]

2 [5 pts] Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, y) + iv(x, y)$. We label the real part $u(x, y)$ the *velocity potential* and the imaginary part $v(x, y)$ the *stream function*. The fluid velocity $\vec{V}$ is given by $\vec{V} = \vec{\nabla} u$. If $f(z)$ is analytic,
(**a**) Show that $df/dz = V_x - iV_y$,
(**b**) Show that $\vec{\nabla} \cdot \vec{V} = 0$ (no sources or sinks),
(**c**) Show that $\vec{\nabla} \times \vec{V} = 0$ (irrotational, nonturbulent flow).

3 [8 pts] (a) [4 pts] If $f(z)$ is analytic and bounded [$|f(z)| \leq M$ for some constant M] for all z, show that $f(z)$ is a constant. This is called Liouville's theorem (complex analysis version. He also has others).
(b) [4 pts] Prove that every polynomial of order $n \geq 1$ with constant coefficients $\in \mathbb{C}$ has at least one root. This is the fundamental theorem of algebra.