Physics 464/511 Homework #3
Due: Oct. 3, 2016 at 5:00 P. M.

1 Consider the unit sphere S^2. Embedded in three dimensional Euclidean space $\mathbb{R}^3$ with cartesian coordinates, this is the set $\{(x, y, z)|x^2+y^2+z^2 = 1\}$, but as it is a two dimensional space, we usually describe it with spherical polar coordinates θ, ϕ, fixing the radius $r = 1$, with $z = \cos\theta$ and $(x, y) = \sin\theta(\cos\phi, \sin\phi)$. The coordinates θ and ϕ are well-defined, however, only for $\theta \neq 0$ and $\neq \pi$. So we have a chart C for $\mathcal{U}$ the unit sphere with the north and south pole removed.

The unit sphere has a natural metric induced from the Euclidean metric of $\mathbb{R}^3$, with which we are very familiar:

$$(ds)^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2, \qquad \text{or} \qquad g_{\theta\theta} = 1, g_{\phi\phi} = \sin^2\theta, g_{\theta\phi} = 0.$$

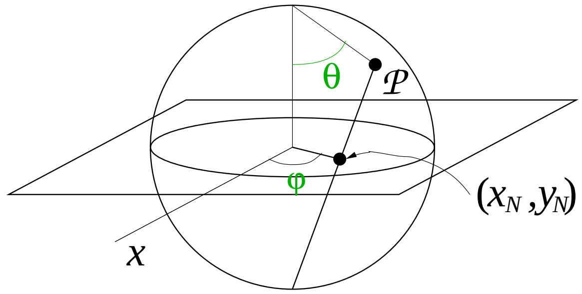
(a) Calculate the Christoffel symbol ${\Gamma^j}_{k\ell}$

(b) The shortest path (on the sphere) between two points is determined by the geodesic equations. Find the equations which determine $\theta(s)$ and $\phi(s)$, where the s parameter is the path length. Show that one of these second order equations is essentially the conservation of L_z.

(c) It is easier to guess the solution from the knowledge that the answer is a great circle, the intersection of the sphere with a plane though the origin, rather than solving the equations from scratch. Show that this is correct, satisfying the geodesic equations.

2 Consider the unit sphere S^2 as the set of point $\{(x, y, z)|x^2+y^2+z^2 = 1\}$ in $\mathbb{R}^3$. This is really a two dimensional manifold, so it needs to be considered as such. Consider the two open sets $\mathcal{U}_N$ which is the sphere minus the South pole, and $\mathcal{U}_S$, the sphere minus the North pole. Define the charts C_N and C_S

by considering the equatorial plane $\mathcal{E}$. For each point $\mathcal{P} \in \mathcal{U}_N$ on the sphere, draw straight lines (in $\mathbb{R}^3$) through the South pole and $\mathcal{P}$, and define $\phi_N(\mathcal{P})$ to be the point (x_N, y_N) at which the line intersects the equatorial plane. Similarly define $\phi_S(\mathcal{P})$ to be the point in equatorial plane on the straight line through the North pole and $\mathcal{P}$, but call that point $(x_S, -y_S)$. [The reason for inverting y is to keep $dx_S \wedge dy_S$ pointing to the outside of the sphere.]



(a) Find the coordinates of $\phi_N(\mathcal{P})$ in terms of the (x, y, z) of $\mathcal{P}$ in $\mathbb{R}^3$, and the inverse relation.

(b) Do the same for $\phi_S(\mathcal{P})$, and find the transition function

$$\phi_{SN} = \phi_S \circ \phi_N^{-1} = \phi_s\left(\phi_N^{-1}(x_N, y_N)\right).$$

(c) The metric on the sphere is $(ds)^2 = (d\theta)^2 + \sin^2\theta (d\phi)^2$. Find the metric on the chart C_N. What is the relation to the ordinary Euclidean metric on the plane?