Physics 464/511 Homework #2
Due: Sept. 26, 2016 at 5:00 P. M.

1 [5 pts] Evaluate

$$\frac{1}{3}\int_S \vec{r}\cdot d\vec{\sigma}$$

over the unit cube defined by the point $(0,0,0)$ and the unit intercepts on the x-, y-, and z-axes. Note that

(a) $\vec{r}\cdot d\vec{\sigma}$ is zero for three of the surfaces and

(b) each of the three remaining surfaces contributes the same amount to the integral.

2 [5 pts] Show that

$$\frac{1}{3}\int_S \vec{r}\cdot d\vec{\sigma} = V$$

where V is the volume enclosed by the closed surface S.

3 [5 pts] An *algebra* consists of a vector space A over a field F, together with a binary operation of multiplication on the set A of vectors, $(A\times A \to A)$ such that for all $a \in F$ and $\alpha, \beta, \gamma \in A$, the following are satisfied:

(A) $(a\alpha)\beta = a(\alpha\beta) = \alpha(a\beta)$

(B) $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$

(C) $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

If, in addition, **(D)**: $\alpha(\beta\gamma) = (\alpha\beta)\gamma$, A is an associative algebra.

(a) Show that linear transformations on a finite-dimensional vector space V into itself (endomorphisms) form an associative algebra A.

(b) Define the commutator of two elements T_1 and T_2 of A by $[T_1, T_2] := T_1T_2 - T_2T_1$. Show that

(i) $[T_1, T_2] = -[T_2, T_1]$.

(ii) $[[T_1, T_2], T_3] + [[T_2, T_3], T_1] + [[T_3, T_1], T_2] = 0$.

Condition (ii) is called the Jacobi identity. Note that it implies that the commutation relation is not an associative multiplication, as in general it implies

$$[[T_1, T_2], T_3] - [T_1, [T_2, T_3]] = [[T_1, T_2], T_3] + [[T_2, T_3], T_1] = -[[T_3, T_1], T_2]$$

which need not be zero.

A vector space L with an additional bilinear operation $L \times L \to L$ satisfying conditions (i) and (ii) is called a *Lie algebra*. The elements need not come originally as a commutator of elements of an associative algebra, as they did here. But from a Lie algebra we can define the *enveloping algebra*, which is associative.

4 [5 pts] [Note: I don't expect you to completely resolve the difficulties here, but I want you to show that you recognize what the issues are, and give indications of how to resolve them.]

In quantum mechanics, the states of a system are taken to be a Hilbert space. For a single particle in one dimension, the wave functions $\psi(x)$ have an inner product given by $\langle \phi | \psi \rangle = \int_{\mathbb{R}} \phi^*(x)\psi(x)\, dx$. The classical degrees of freedom such as position x and momentum p are replaced by linear operators $\mathbf{x}$ and $\mathbf{p}$ or combinations of these, and the reality of these variables classically become conditions that the operators are *hermitian*, which means equal to its *adjoint*. The adjoint of a linear operator $\mathbf{A}$ is a linear operator $\mathbf{A}^\dagger$ such that $\langle \phi | \mathbf{A}\psi \rangle = \left\langle \mathbf{A}^\dagger \phi \middle| \psi \right\rangle$ for any wave functions ψ and ϕ.

The expected value of a classical quantity represented by $\mathbf{A}$ in a state represented by a normalized wavefunction ψ is given by $\langle A \rangle = \langle \psi | \mathbf{A}\psi \rangle = \int_{\mathbb{R}} \psi^* \mathbf{A} \psi \, dx$. Its complex conjugate is $\langle A \rangle^* = \langle \mathbf{A}\psi | \psi \rangle = \left\langle \psi \middle| \mathbf{A}^\dagger \psi \right\rangle$, so it is real if $\mathbf{A}$ is hermitian. The momentum operator is $\mathbf{p} = \dfrac{\hbar}{i}\dfrac{\partial}{\partial x}$, and then the commutator $[\mathbf{x}, \mathbf{p}] := \mathbf{xp} - \mathbf{px}$ acting of a wave function $|\psi\rangle$ gives

$$x \frac{\hbar}{i}\frac{\partial \psi}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x} x\psi = i\hbar\psi \qquad \text{so} \quad [\mathbf{x}, \mathbf{p}] = i\hbar$$

and $\mathbf{x}$ and $\mathbf{p}$ are canonically conjugate variables.

Clearly $\mathbf{x}$ is hermitian, but how is it that $\mathbf{p}$ is? That is, what have we been sloppy about in specifying our space of wave functions that needs specifying to verify that $\mathbf{p} = \mathbf{p}^\dagger$?

Perhaps it will be illuminating to consider instead the azimuthal angle ϕ and the z component of angular momentum, $L_z = \dfrac{\hbar}{i}\dfrac{\partial}{\partial\phi}$ with $[L_z, \phi] = i\hbar$. These have the same commutation relation as $\mathbf{p}$ and $\mathbf{x}$. According to the Heisenberg uncertainty principle[1], if $[\mathbf{A}, \mathbf{B}] = i\mathbf{C}$, then the uncertainty in A, given by $(\Delta A)^2 = \int \psi^* \left(A - \langle A\rangle\right)^2 \psi\, dx$, and the uncertainty in B are restricted by $\Delta A\, \Delta B \geq \frac{1}{2}|\langle C\rangle|$. So $\Delta L_z\, \Delta\phi \geq \hbar/2$, but we can have states of definite L_z, so $\Delta L_z = 0$, while $-\pi \leq \phi \leq \pi$, so $\Delta\phi \leq \pi$. How can that be?

[1]Proof: We need to use the Schwarz inequality:

$$||\phi||^2 ||\psi||^2 \geq |\langle\phi|\psi\rangle|^2.$$

As $(\Delta A)^2 = ||(A - \langle A\rangle\,)\psi||^2$ the Schwarz inequality tells us

$$\begin{aligned}(\Delta A)^2\,(\Delta B)^2 &\geq |\langle (A - \langle A\rangle\,)\psi|(B - \langle B\rangle\,)\psi\rangle| = \langle\psi|(A - \langle A\rangle\,)\,(B - \langle B\rangle\,)\psi\rangle \\ &= \frac{1}{2}\Big\langle\psi\ \Big\|\Big[(A - \langle A\rangle\,)\,(B - \langle B\rangle\,) + (B - \langle B\rangle\,)\,(A - \langle A\rangle\,) + [A, B]\Big]\psi\Big\rangle|.\end{aligned}$$

The first two terms are hermitian conjugates so give a real value, while the commutator is i times the hermitian operator C, so gives an imaginary, and the absolute value must be $\geq$ the imaginary part.