Problem set VIII “Kepler problem in GR”

Due April 4, 2022

Problem I

We are interested in the orbit equation, giving the radial coordinate in terms of the azimuthal angle, \( r = r(\phi) \).

(a) First Classical Mechanics. For a particle moving in a Newtonian central potential \( V = -\frac{GmM}{r} \), \( m \ll M \) derive the Binet equation

\[
u'' + u = \frac{m^2 GM}{L^2},
\]

where \( u = \frac{1}{r} \) and \( L \) is the value of the angular momentum, which is conserved. Solve the equation and verify that the solutions are ellipses.

(b) Now GR. Using the static Schwarzschild metric derive the modified equation for \( u \):

\[
u'' + u = \frac{m^2 GM}{L^2} + 3 GM u^2.
\]

Show that the additional term is small for Mercury (look up the orbital parameters as needed).

(c) Now solve this using perturbation theory to lowest order in the perturbation (the additional term). Give your answer as an expression for \( u(\phi) \) in terms of the eccentricity, \( e \), of the unperturbed ellipse, \( GM \) and \( L \).

(d) Calculate the perihelion advance \( \Delta \phi \) per revolution in terms of the eccentricity and the semi-major axis, \( a \), of the unperturbed ellipse. Plug numbers, and convert your answer to the shift per century on Earth (rather than per revolution).

Planets also affect the orbit of Mercury and contribute to the shift of the perihelion. The magnitude of the planetary effect is about 12 times larger than the GR effect, but it was already calculated with great precision in the time of Einstein. It is known today to better than 6 parts in \( 10^4 \). So the GR-effect, unknown pre-Einstein, was a considerable anomaly, about a thousand times larger than the error of the calculation.
Problem II

A particle of mass $m$ is in a stable circular orbit of radius $R > r_g$ in the Schwarzschild gravitational field:

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)(dt)^2 + \left(1 - \frac{r_g}{r}\right)^{-1} (dr)^2 + r^2 (d\theta)^2 + \sin^2 \theta (d\phi)^2 .$$

(a) Compute the orbital period as indicated on a clock attached to the particle.

(b) At the completion of each orbit the particle emits a photon. This is eventually received by an observer $O$ located very far away. What is the period as perceived by $O$?

(c) An observer $O'$ is fixed at radius $R$ on the orbit, held there by a rocket motor. What is the period of the particle as measured by $O'$?

Problem III

The spin of a gyroscope is characterized by a classical spin four-vector $S^\mu$ satisfying $S_\mu u^\mu = 0$ (where $u^\mu = \frac{dx^\mu}{d\tau}$ is the center-of-mass four-velocity of the gyroscope) and $S^\mu S_\mu = \text{constant}$ (see Lecture notes 8). If the gyroscope moves along a geodesic orbit, the spin vector evolves by parallel transport. Consider the case of a gyroscope moving in a stable circular geodesic orbit of radius $R$ in a Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)(dt)^2 + \left(1 - \frac{r_g}{r}\right)^{-1} (dr)^2 + r^2 d\Omega^2 .$$

a) Convert the parallel transport equation into an equation for the evolution, as a function of azimuthal angle around the orbit, of the spatial components of the spin.

b) Solve this equation to find the rate of precession of the spin vector as seen by an observer fixed at a given point on the orbit.