Problem set VI “Linear approximation”
Due March 10, 2022

Problem I
A good approximation to the metric outside the surface of the Earth is provided by
\[ ds^2 \approx -\left(1 - \frac{r_g}{r}\right)(dt)^2 + \left(1 + \frac{r_g}{r}\right)\left[(dr)^2 + r^2((d\theta)^2 + \sin^2(\theta)(d\phi)^2)\right], \]
where \( r_g = 2GM \):

(a) Imagine a clock on the surface of the Earth at a distance \( R_1 \) from the Earth’s center, and another clock on a tall building at a distance \( R_2 \) from the Earth’s center. Calculate the time elapsed on each clock as a function of the coordinate time \( t \). Which clock moves faster?

(b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth. You can work to first order in \( r_g/r \) if you like. What is the orbital period?

(c) How much proper time elapses while a satellite at radius \( R \) (skimming along the surface of the earth, neglecting air resistance) completes one orbit? Plug in the actual numbers for the radius of the Earth and so on (don’t forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on a clock that is stationary on the surface?

Problem II
For the metric from Problem I (weak static gravitational field)
\[ g_{00} = -\left(1 - \frac{r_g}{r}\right), \quad g_{ij} = \delta_{ij}\left(1 + \frac{r_g}{r}\right), \quad g_{0i} = 0 \]
where \( r_g = 2GM \):

(a) Calculate all components of the Riemann tensor \( R_{[\mu\nu][\lambda\rho]} \).

(b) Write down the geodesic deviation equation in a locally inertial frame. Assume that all velocities are \( \ll c \).

(c) Solve the geodesic deviation equation from part (b) for a space station making circular orbit of radius \( R \) around the Earth.

(d) Suppose a Skylab satellite orbits the Earth in a circular equatorial orbit. An astronaut jettisons a bag of garbage into a nearby orbit and watches it move relative to the satellite. At a given time the separation of the Skylab and its garbage is described by the vector
\[ \xi^i = x^i(\text{garbage}) - x^i(\text{skylab}) . \]
Find the components of the relative motion \( \xi^i \) as a function of time.
Problem III

A non-relativistic particle of mass $m = 1$ in $\mathbb{R}^3$ is constrained to move on the ellipsoid surface

$$\Phi(x, y, z) \equiv \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 - 1 = 0.$$ 

(a) Using the Newton equation of motion show that

$$I = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) \left\{ \left(\frac{\dot{x}}{a}\right)^2 + \left(\frac{\dot{y}}{b}\right)^2 + \left(\frac{\dot{z}}{c}\right)^2 \right\} = \frac{(\vec{\nabla}\Phi)^2}{8} \left(\vec{r} \cdot \frac{d\vec{\nabla}\Phi}{dt}\right)$$

is a conserved quantity (Joachimsthal-Jacobi integral of motion).

(b) Show that the Lagrangian (which coincides with the (kinetic) energy in this case) and Joachimsthal-Jacobi integral of motion from (a) are expressed in terms of the ellipsoidal coordinates as

$$L = \frac{1}{8} \left( \frac{\mu(\mu - \nu) \dot{\mu}^2}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} + \frac{\nu(\nu - \mu) \dot{\nu}^2}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \right)$$

$$I = -\frac{1}{4abc^2} \left( \frac{\mu \nu(\mu - \nu) \dot{\mu}^2}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} + \frac{\mu \nu(\nu - \mu) \dot{\nu}^2}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \right)$$

(c) Express the Hamiltonian $H$ and the Joachimsthal-Jacobi integral $I$ in terms of the canonical variables ($\mu, \nu, p_\mu, p_\nu$) and show that

$$2\mu H + (abc)^2 I = 4 \left( a^2 + \mu \right) \left( b^2 + \mu \right) \left( c^2 + \mu \right) \frac{p_\mu^2}{\mu}$$

$$2\nu H + (abc)^2 I = 4 \left( a^2 + \nu \right) \left( b^2 + \nu \right) \left( c^2 + \nu \right) \frac{p_\nu^2}{\nu}.$$ 

Also check that the Poisson bracket $\{ I, H \}$ vanishes.

(d) Using the Hamilton-Jacobi method reduce the Jacobi’s Geodesic problem to quadratures.

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1 This problem is optional. It will not be graded.

2 The problem of computing the geodesics on an ellipsoid is a classic problem. For a two-dimensional ellipsoid, its solution was announced by Jacobi on the 28th of December, 1838. Using the remarkable substitution, he reduced this problem to quadratures. The result was published in the paper [C. Jacobi, JRAM 19, 1839, pp. 309-313].
Problem IV

Find the shape of an ordinary optical lens that would imitate the deflection of light rays by the weak gravitational field of a star.