Problem set V “Riemann tensor”

Due to February 28, 2022

Problem I

(a) For a two dimensional manifold express the Riemann tensor $R_{[\mu\nu][\lambda\rho]}$ in terms of the scalar curvature $R = g^{\mu\lambda} g^{\nu\rho} R_{[\mu\nu][\lambda\rho]}$.

(b) For the diagonal metric of the form

$$ds^2 = (H_1(x))^2 (dx^1)^2 + (H_2(x))^2 (dx^2)^2$$

find the scalar curvature.

(c) Consider the ellipsoid which is defined in $\mathbb{E}^3$ by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 :$$

Calculate the scalar curvature of this 2D manifold w.r.t the metric induced by the Euclidean metric of $\mathbb{E}^3$. Express the result in terms of the Cartesian coordinates $(x, y, z)$.

**Hint:** You can use the ellipsoidal coordinates from Problem II (HW2).
Problem II

Let \( x^i(\sigma), i = 1, 2, \) be a piecewise smooth, closed curve bounding a region \( U \) of a smooth surface in Euclidean 3-space. Show that\(^1\)

\[
\varphi = \int_U K \sqrt{g} \, d^2x, \quad K = \frac{1}{2} R
\]

(here \( R \) is the scalar curvature w.r.t the metric induced by the Euclidean metric of \( \mathbb{E}^3 \)) is the angle through which a vector rotates after being parallel-transported around the curve \( x^i(\sigma) \).

\(^1\)\( K = \frac{1}{2} R \) is called the Gaussian curvature of 2D surface.
Problem III

(a) Let \( M \) be the connected part of the two sheet hyperboloid in \( \mathbb{E}^3 \) defined by the conditions

\[
M : \quad -\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1, \quad z > 0:
\]

Write down the metric on this manifold induced by the standard Euclidean metric of \( \mathbb{E}^3 \) and calculate the curvature \( R \). As in Problem II (HW4) use the polar coordinates \((r, \phi)\) of the orthogonal projection on the \( xy \)-plane. Check that the curvature at the tip of the hyperboloid corresponding to \( r = 0 \) is given by

\[
R_0 = \frac{2 \nu^2}{a^2}, \quad \text{where} \quad \nu = \frac{c}{a}.
\]

Also show that

\[
\int_M d^2x \sqrt{g} R = 4\pi (1 - \beta), \quad \text{where} \quad \beta = \frac{1}{\sqrt{1 + \nu^2}}.
\]

(b) Show that with a proper choice of the coordinates the induced metric on the two sheet hyperboloid from (a) can be brought to the form

\[
\text{ds}_{\text{hyp}}^2 = (d\rho)^2 + \left[ \rho f(\rho/a) \right]^2 (d\phi)^2 \quad (0.1)
\]

with \( 0 \leq \rho < \infty \), \( \phi \) is an angular variable, \( \phi \sim \phi + 2\pi \), and \( f(u) \) is a monotonically decreasing function such that

\[
f(u) = \begin{cases} 
1 + O(u^2) & \text{as } u \to 0 \\
\beta + O(u^{-2}) & \text{as } u \to \infty
\end{cases}
\]
Notice that when \( a, c \to 0 \) assuming the parameter \( \nu = \frac{c}{a} \) is kept fixed, the connected component of the two sheet hyperboloid becomes a cone (which is not a smooth manifold!)

\[
z = \nu \sqrt{x^2 + y^2}:
\]

In this limit, the metric \([0.1]\) becomes

\[
ds_{\text{hyp}}|_{a \to 0} \to ds_{\text{cone}}^2 = (d\rho)^2 + \beta^2 \rho^2 (d\phi)^2.
\]

(c) Calculate the curvature \( R \) of the surface of a cone equipped with the metric \( ds_{\text{cone}}^2 \) above with \( 0 < \rho < \infty \), \( \phi \) is an angular variable, \( \phi \sim \phi + 2\pi \) and \( \beta \) is a parameter.

**Hint:** Strictly speaking, \( ds_{\text{cone}}^2 \) is not a smooth metric because of the presence of the tip. One may start with smooth metric with \( \beta^2 \rho^2 \) replaced by \( \beta^2_c(\rho) \rho^2 \), with \( \beta_c(\rho) \to \beta \) at \( \rho \gg \epsilon \), and \( \beta_c(\rho) \to 1 \) as \( \rho \to 0 \), and then send \( \epsilon \to 0 \). The hyperboloid metric \([0.1]\) is a particular example of the smooth regularized metric with \( \beta_c(\rho) \) taken to be \( f(\rho/a) \) and \( a \) plays the role of the regularization parameter \( \epsilon \).

(d) Find the angle of rotation resulting from parallel transport of a vector along a closed curve which goes around the tip of the cone \( \rho = 0 \). Explain why the result agrees with your findings in part (d).
Problem IV

Consider space-time with the metric

\[ d\tau^2 = L \frac{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}{z^2} \]

where \( L \) is a parameter.

(a) Calculate all 20 components of the Riemann curvature tensor. What is the covariant derivative \( \nabla_\sigma R_{\mu\nu\lambda\delta} \)? Find the Ricci tensor and the scalar curvature \( R \).

(b) Find geodesic curves (space- and time-like).