Problem I

Suppose we are given two different orthogonal coordinate systems which share the origin in common. Let $\vec{e}_i$ and $\vec{e}_i'$ ($i = 1, 2, 3$) be unit vectors along the coordinate axes $OXYZ$ and $OX'Y'Z'$, respectively,

$$\vec{e}_i \cdot \vec{e}_j = \vec{e}_i' \cdot \vec{e}_j' = \delta_{ij}.$$ 

Introduce the transformation $\hat{S}$ as a linear operator defined by the conditions,

$$\vec{e}_i' = \hat{S} \ast \vec{e}_i \quad (i = 1, 2, 3).$$ 

Such transformations are known as orthogonal. The vectors $\vec{e}_i'$ can be linearly expressed in terms of $\vec{e}_i$,

$$\vec{e}_i' = \vec{e}_j S^j_i,$$

One can think of the unit vectors $\vec{e}_i$ as columns

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Then the position vector would be given by

$$\vec{r} \equiv \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \sum_{i=1}^{3} x^i \vec{e}_i \equiv x^i \vec{e}_i'.$$

The same vector can be re-written in the other basis $\vec{r}' = x'^j \vec{e}'_j$ with $x'^j = S^j_i x^i$. 

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1One can think of the unit vectors $\vec{e}_i$ as columns
where the $3 \times 3$ matrix for the operator $\hat{S}$,

$$
\mathbf{S} = \begin{pmatrix}
S_1^1 & S_1^2 & S_1^3 \\
S_2^1 & S_2^2 & S_2^3 \\
S_3^1 & S_3^2 & S_3^3
\end{pmatrix},
$$

satisfies the condition

$$
\mathbf{S}^T \mathbf{S} = 1
$$

(the superscript “$T$” denotes the matrix transposition). For an arbitrary orthogonal transformation

$$
\det^2(\mathbf{S}) = 1, \quad \text{i.e.} \quad \det(\mathbf{S}) = \pm 1.
$$

An orthogonal transformation with determinant $+1$ is said to be proper.

(a) Show that the linear operator $\hat{S}$ such that

$$
\hat{\mathbf{e}}_1' = \hat{S} \cdot \hat{\mathbf{e}}_1 = \frac{1}{4} \hat{\mathbf{e}}_1 + \frac{1 + 2\sqrt{2}}{4} \hat{\mathbf{e}}_2 - \frac{2 - \sqrt{2}}{4} \hat{\mathbf{e}}_3,
$$

$$
\hat{\mathbf{e}}_2' = \hat{S} \cdot \hat{\mathbf{e}}_2 = \frac{1 - 2\sqrt{2}}{4} \hat{\mathbf{e}}_1 + \frac{1}{4} \hat{\mathbf{e}}_2 + \frac{2 + \sqrt{2}}{4} \hat{\mathbf{e}}_3,
$$

$$
\hat{\mathbf{e}}_3' = \hat{S} \cdot \hat{\mathbf{e}}_3 = \frac{2 + \sqrt{2}}{4} \hat{\mathbf{e}}_1 - \frac{2 - \sqrt{2}}{4} \hat{\mathbf{e}}_2 + \frac{1}{2} \hat{\mathbf{e}}_3
$$

is a proper orthogonal transformation.

(b) According to Euler’s theorem an arbitrary proper orthogonal transformation is a rotation.

Illustrate this statement using the transformation from (a), i.e., determine the unit vector $\mathbf{n}$ along the corresponding axis of rotation and the rotation angle $\phi$.

(c) For an arbitrary proper orthogonal transformation express the matrix elements $S_i^j$ in terms of the unit vector along the axis of rotation $\mathbf{n}$ and the rotation angle $\phi$.

Hint: To simplify the final expression it is convenient to use the so-called Euler parameters:

$$
\rho_0 = \cos(\phi/2), \quad \rho_i = \sin(\phi/2) \ n_i \quad (i = 1, 2, 3).
$$
**Problem II**

A cart rolls on a long table with velocity \( v \). A smaller cart rolls on the first cart in the same direction with velocity \( v \) relative to the first cart. A third cart rolls on the second cart in the same direction with velocity \( v \) relative to the second cart, and so on up to \( n \) carts. What is the velocity \( v_n \) of the \( n^{th} \) cart in the frame of the table? What does \( v_n \) tend to as \( n \to \infty \)?

**Problem III**

Consider an inertial frame \( S \) with coordinates

\[
    x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \text{(in the units } c = 1) \]

and a frame \( S' \) with coordinates \( x'^\mu \) related to \( S \) by a boost with velocity \( \vec{V} = \begin{pmatrix} 0 \\ V \\ 0 \end{pmatrix} \) along the \( y \)-axis (we assume \( V > 0 \)). Imagine there is a wall at rest in \( S' \), lying along the line \( x' = +y' \).

(a) Suppose there is a ball traveling in the \( xy \)-plane that elastically hits the wall. In the co-ordinate frame \( S \), what is the relationship between the incident velocity \( \vec{v}_\text{in} = \begin{pmatrix} v_x^\text{in} \\ v_y^\text{in} \\ 0 \end{pmatrix} \) and the outgoing velocity \( \vec{v}_\text{out} = \begin{pmatrix} v_x^\text{out} \\ v_y^\text{out} \\ 0 \end{pmatrix} \) of the ball.

(b) Find the incident angle \( \theta_{\text{in}} \) of the ball and the reflected angle \( \theta_{\text{out}} \). Calculate numerical values of \( \theta_{\text{in}} \) and \( \theta_{\text{out}} \) for

\[
    v_x^\text{in} = -0.5, \quad v_y^\text{in} = +0.3, \quad V = 0.2
\]

![Diagram of the problem](image)
Problem IV

(a) Show that the completely antisymmetric Levi-Civita symbol

\[ \varepsilon_{\mu\nu\lambda\rho} = \begin{cases} +1 & \text{if } (\mu, \nu, \lambda, \rho) \text{ is an even permutation of } (0, 1, 2, 3) \\ -1 & \text{if } (\mu, \nu, \lambda, \rho) \text{ is an odd permutation of } (0, 1, 2, 3) \\ 0 & \text{otherwise} \end{cases} \]

is a tensor of rank \((0, 4)\) w.r.t. the proper Lorentz transformations. The latter are defined by the matrix

\[ \Lambda = \begin{pmatrix} \Lambda_0^0 & \Lambda_0^1 & \Lambda_0^2 & \Lambda_0^3 \\ \Lambda_1^0 & \Lambda_1^1 & \Lambda_1^2 & \Lambda_1^3 \\ \Lambda_2^0 & \Lambda_2^1 & \Lambda_2^2 & \Lambda_2^3 \\ \Lambda_3^0 & \Lambda_3^1 & \Lambda_3^2 & \Lambda_3^3 \end{pmatrix} : \quad \Lambda^T \eta \Lambda = \eta, \quad \det(\Lambda) = 1, \]

where

\[ \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \]

(b) Show that

\[ \varepsilon^{\alpha\beta\gamma\sigma} \varepsilon_{\mu\nu\lambda\rho} = -\det \begin{pmatrix} \delta^\alpha_\mu & \delta^\alpha_\nu & \delta^\alpha_\lambda & \delta^\alpha_\rho \\ \delta^\beta_\mu & \delta^\beta_\nu & \delta^\beta_\lambda & \delta^\beta_\rho \\ \delta^\gamma_\mu & \delta^\gamma_\nu & \delta^\gamma_\lambda & \delta^\gamma_\rho \\ \delta^\sigma_\mu & \delta^\sigma_\nu & \delta^\sigma_\lambda & \delta^\sigma_\rho \end{pmatrix} \]

where \(\varepsilon^{\alpha\beta\gamma\sigma} \equiv \eta^{\alpha\mu} \eta^{\beta\nu} \eta^{\gamma\lambda} \eta^{\sigma\rho} \varepsilon_{\mu\nu\lambda\rho}\) and \(\delta^\alpha_\mu\) is the Kronecker delta symbol:

\[ \delta^\alpha_\mu = \begin{cases} 0, & \text{if } \mu \neq \alpha \\ 1, & \text{if } \mu = \alpha \end{cases} \]
Problem V

Electric and magnetic fields transform under Lorentz transformations as the components of an anti-symmetric tensor

\[ F_{\mu\nu} = -F_{\nu\mu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \]  

(0.1)

i.e.,

\[ F_{0i} = -E_i, \quad F_{ij} = \varepsilon_{ijk} B_k \quad (i,j,k \in \{1,2,3\}) \]

(a) Show that \( \vec{B}^2 - \vec{E}^2 \) and \( \vec{E} \cdot \vec{B} \) are invariant under the proper Lorentz transformations (which are all possible compositions of space rotations and Lorentz boosts). Are there any invariants which are not merely algebraic combinations of these two?

(b) Given constant \( \vec{E} \) and \( \vec{B} \) in one frame and assuming that \( \vec{E} \cdot \vec{B} \neq 0 \), find another frame in which \( \vec{E}' \) and \( \vec{B}' \) are parallel to each other (i.e. determine the direction and magnitude of the relative velocity \( \vec{V} \) of the two frames).

(c) Let the constant fields \( \vec{E} \) and \( \vec{B} \) be such that \( \vec{E} \perp \vec{B} \). Assuming that \( |\vec{E}| > |\vec{B}| \) find the frame in which \( \vec{B}' = 0 \). Similarly for \( |\vec{E}| < |\vec{B}| \) find the frame where \( \vec{E}' = 0 \).

(c) Show that if \( \vec{E} \perp \vec{B} \) and \( |\vec{E}| = |\vec{B}| \), then the same relations hold true in any inertial frame.