Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry*

F. K. Manasse† and C. W. Misner‡
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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Fermi coordinates, where the metric is rectangular and has vanishing first derivatives at each point of a curve, are constructed in a particular way about a geodesic. This determines an expansion of the metric in powers of proper distance normal to the geodesic, of which the second-order terms are explicitly computed here in terms of the curvature tensor at the corresponding point on the base geodesic. These terms determine the lowest-order effects of a gravitational field which can be measured locally by a freely falling observer. An example is provided in the Schwarzschild metric. This discussion of Fermi Normal Coordinate provides numerous examples of the use of the modern, coordinate-free concept of a vector and of computations which are simplified by introducing a vector instead of its components. The ideas of contravariant vector and Lie Bracket, as well as the equation of geodesic deviation, are reviewed before being applied.

I. INTRODUCTION

In 1922 Fermi showed[1] that, given any curve in a Riemannian manifold, it is possible to introduce coordinates near this curve in such a way that the Christoffel symbols vanish along the curve, leaving the metric there rectangular. Several developments of this idea followed. One was a generalization[2] of the theorem to a manifold with a symmetric affine connection $\Gamma_{\mu\nu}{}^{\sigma}$, but without necessarily assuming any metric structure. A second development was an inquiry which showed that in general no coordinates exist for which $\Gamma_{\mu\nu}{}^{\sigma} = 0$ on surfaces of dimension greater than one, and which developed criteria for the special situations where this was possible.[3] A third variation of Fermi's idea is the set of coordinates based on an arbitrary curve which Synge[4] calls Fermi coordinates. Here one allows a few nonzero Christoffel symbols, although retaining a rectangular metric, for the advantage of making the curve become an axis of the coordinate system. These coordinates, as Synge shows, form a nonrotating system in a natural physical sense for a (not necessarily freely falling) observer in a gravitational field.

In this paper we consider not a modification or generalization of Fermi's idea, but a specialization and particularization of it. We specialize to the case where the curve in question is a geodesic, and we choose a particular set out of the many coordinate systems which satisfy his $\Gamma_{\mu\nu}{}^{\sigma} = 0$ condition along this geodesic. The resulting coordinates we call *Fermi normal coordinates* because of an analogy to that particular choice of the many coordinates satisfying $\Gamma_{\mu\nu}{}^{\sigma} = 0$ at a single point, called Riemann normal coordinates,[5] which in addition gives the series expansion[6]

$$ds^2 = \{\eta_{\mu\nu} + \tfrac{1}{3}R_{\alpha\mu\beta\nu}x^{\alpha}x^{\beta} + O[(x)^3]\}\, dx^{\mu}\, dx^{\nu}. \qquad (1)$$

The primary mathematical contribution of this paper is to compute the quadratic terms of a corresponding expansion in Fermi normal coordinates [see Eqs. (66)]. In this case the expansion parameter is the geodesic distance normal to the given geodesic; the expansion is valid for a limited region of space, and for *all time*. Thus, Fermi normal coordinates provide a standardized way in which a freely falling observer can report observations and local experiments. In particular, the quadratic terms of the metric, which we compute in terms of the curvature,

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† Communications Development Training Program Fellow from Bell Telephone Laboratories.

‡ Alfred P. Sloan Research Fellow.

[1] E. Fermi, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fiz. Mat. Nat. **31**, 21, 51 (1922).

[2] L. P. Eisenhart, *Non-Riemannian Geometry*, (American Mathematical Society Colloquium Publications, New York, New York, 1927), Sec. 25. The Fermi normal coordinates developed in the present paper are also defined in (symmetric) affine spaces, and all our results which can be stated in affine spaces are valid there. The proofs are obtained by replacing every set of orthonormal vectors by a set of linearly independent vectors.

[3] L. O'Raifeartaigh, Proc. Roy. Irish Acad. **A59**, 2 (1958).

[4] J. L. Synge, *Relativity, The General Theory* (North-Holland Publishing Company, Amsterdam, 1960).

[5] See, for example, L. P. Eisenhart, *Riemannian Geometry*, (Princeton University Press, Princeton, New Jersey, 1926).

[6] Here $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$ is the Lorentz metric. We shall use Greek indices for space-time (μ, ν, etc. = 0, 1, 2, 3), while Latin indices give components along spatial axes (i, j, etc. = 1, 2, 3). Our sign conventions for the curvature tensor are

$$R_{\mu}{}^{\nu}{}_{\alpha\beta} = \partial_{\alpha}\Gamma_{\mu}{}^{\nu}{}_{\beta} - \partial_{\beta}\Gamma_{\mu}{}^{\nu}{}_{\alpha} - (\Gamma_{\mu}{}^{\sigma}{}_{\alpha}\Gamma_{\sigma}{}^{\nu}{}_{\beta} - \Gamma_{\mu}{}^{\sigma}{}_{\beta}\Gamma_{\sigma}{}^{\nu}{}_{\alpha}),$$

and

$$R_{\mu\nu} = R_{\mu}{}^{\alpha}{}_{\alpha\nu}.$$

The Riemann tensor convention corresponds to Cartan's definition (reference 13) of the curvature forms $O_{\mu}{}^{\nu} = \frac{1}{2}R_{\mu}{}^{\nu}{}_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$ in terms of the connection forms

$$\omega_{\mu}{}^{\nu} = \Gamma_{\mu}{}^{\nu}{}_{\alpha}\, dx^{\alpha},\ \theta_{\mu}{}^{\nu} = d\omega_{\mu}{}^{\nu} - \alpha_{\mu}{}^{\sigma}\,\omega_{\sigma}{}^{\nu},$$

which definition is also valid in orthogonal (or other nonholonomic) frames.

determine the effects of gravitational field gradients upon experiments done in a freely falling elevator.

The procedure for constructing Fermi normal coordinates, which is given in Sec. II, is a variation of the standard procedure for constructing Riemann normal coordinates. It is also a special case of the procedures used by Levi-Civita[7] or Synge[4] to construct (inequivalent versions of) Fermi coordinates about an arbitrary, nongeodesic, curve. The present paper is very closely related to Levi-Civita's, since it discusses some of the same topics, but in inverse order. Levi-Civita, in the paper in question, developed for the first time the equation of geodesic deviation and used Fermi coordinates as a technique for simplifying this equation to display its properties more clearly. In contrast, our primary interest is here in the Fermi coordinates, but we shall use the equation of geodesic deviation as a device for studying the properties of Fermi coordinates and for computing the metric tensor in these coordinates.

The major part of the present paper is devoted to studying properties of the Fermi normal coordinates constructed in Sec. II. In Sec. IV we show that this construction leads to a nonsingular coordinate system in a neighborhood of the given geodesic, and in Sec. V we show that these coordinates satisfy the Fermi conditions

$$g_{\mu\nu}|_G = \eta_{\mu\nu}, \tag{2}$$

$$\Gamma_\mu{}^\alpha{}_\nu|_G = 0, \tag{3}$$

along the given geodesic G. In these discussions, as well as in later examples, it is useful to have unambiguous ways of indicating a vector without specifying a coordinate system, and of displaying its components in different coordinate systems without confusion. These notations, based on the idea of a tangent vector as a differentiation, are reviewed in Sec. III. This idea of a vector is also used in Sec. VI where we review the equation of geodesic deviation in order to see precisely what vector satisfies it. Then, in Sec. VII, we note that certain vectors occurring in the construction of Fermi normal coordinates must satisfy the equation of geodesic deviation; using this fact we evaluate the quadratic (curvature) terms in the expansion of the metric analogous to Eq. (1). Finally, Sec. VIII is an example, where, starting from the Schwarzschild metric in standard Schwarzschild coordinates, we evaluate the metric in Fermi normal coordinates surrounding a radial timelike geodesic. This represents this metric in a rest frame of a particle of negligible mass freely falling toward a large central mass. In the following paper,[8] this serves as a starting point from which to compute the metric surrounding a finite but small mass falling radially toward a large central mass, a particular case of the two-body problem in general relativity.

The paragraphs of mathematical "review" (Secs. III and VI), although they contain nothing new or original, are not considered by the authors as the least important part of this paper. Most physicists, even those very familiar with general relativity, continue to use the same definition of a vector as did Einstein, in spite of the considerable progress by mathematicians in the intervening half century. A particularly careful statement of this definition by Synge and Schild[9] gives a precise meaning to the sentence "The quantities v^μ are components of a contravariant vector" without finding it worth the trouble to write a sentence of the form "a contravariant vector is a ..." The end of this sentence is, in fact, either rather unhelpful[10] or rather long[11] when it merely elucidates the transformation law definition. The transformation law outlook on geometry was an attempt to broaden[12] the Erlanger Programm viewpoint: (a geometry is characterized by invariance under a group of transformations) without repudiating it completely. The more geometrical approach to geometry, based on an intuition rooted in the classical studies of curves and surfaces in Euclidean three space, was hampered for a time because its most powerful computational techniques[13] employed elements which were defined only by their intuitive significance. As a consequence, many demonstrations were clear only to mathematicians with sufficient intuition.[13] This difficulty was eliminated by Chevalley[14] who gave new definitions of tangent vectors and differentials, providing them with a

[7] T. Levi-Civita, Math. Ann. 97, 291 (1926).

[8] F. K. Manasse, J. Math. Phys. 4, 746 (1963) (following paper).

[9] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1952), Sec. 1.3.

[10] T. Y. Thomas, *The Differential Invariants of Generalized Spaces* (Cambridge University Press, New York, 1934,) p. 30.

[11] N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, New Jersey, 1951), Sec. 6.4.

[12] O. Veblen and J. H. C. Whitehead, *The Foundations of Differential Geometry*, (Cambridge University Press, New York, 1932, reprinted 1953), Sec. 16.

[13] E. Cartan, *Leçons sur la géométrie des espaces de Riemann* (Gauthier-Villars, Paris, 1951).

[14] C. Chevalley, *Theory of Lie Groups.* (Princeton University Press, Princeton, New Jersey, 1946), p. 77. A definition suitable for differentiable, rather than analytic, manifolds can be found in reference 15 or in H. Flanders, Trans. Am. Math. Soc. 75, 311 (1953). A definition of differentiable manifold which parallels Chevalley's for the analytic case is given by de Rham, *Variétes Différentiables* (Hermann et Cie., Paris, 1955).

clear formal structure compatible with their intuitive significance. These definitions are now also available in introductory texts.[15]

In Sec. III and subsequently we use this definition of a contravariant vector as a tangent to a curve. Since a curve is easily thought of in a coordinate-independent way as a moving point $P(t)$ in the manifold, this approach can avoid all mention of coordinates in defining a vector. A theoretical physicist should not be surprised that the tangent to a curve $P(t)$ is conceived of as the generator of infinitesimal translations along the curve and, hence, denoted by $\partial/\partial t$.

II. CONSTRUCTION OF FERMI NORMAL COORDINATES

Conditions (2) and (3), which Fermi coordinates satisfy, state that to the maximum extent possible one desires space in the neighborhood of some given geodesic G to look like flat space in rectangular coordinates. As motivation for the construction which will be given in this section, we suggest that a plausible way to try to achieve this is to use as many "straight" lines (geodesics) as possible in laying out the coordinates. What follows now is merely a recipe which purports to construct a coordinate system; the proof that it does so (i.e., that the coordinates constructed by this recipe are non-singular) is deferred to Sec. IV. That Eqs. (2) and (3) are satisfied is not shown until Sec. V.

In order to uniquely specify a set of Fermi normal coordinates it is necessary to choose arbitrarily a point P_0 to be the origin, and an orthonormal set of vectors $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ at P_0 to fix the coordinate axes there. The first step in the construction is then to solve the geodesic equation and obtain that unique geodesic G which starts at P_0 with tangent $\mathbf{e}_0$ there. We will describe the geodesic G by the equation

$$P = h(\tau). \tag{4}$$

The condition that G "starts at P_0" is just

$$P_0 = h(0), \tag{5}$$

and does not imply that we refuse to consider negative values of τ.

Because G is a geodesic, its tangent at any two points on G is related by parallel displacement along G. At P_0, the tangent was $\mathbf{e}_0$, which we now call $\mathbf{e}_0(0)$, while $\mathbf{e}_0(\tau)$ will mean the tangent to G at $P = h(\tau)$. Similarly, we can define $\mathbf{e}_i(\tau)$ for $i = 1, 2, 3$ as vectors at $h(\tau)$ obtained by parallel displacement

[15] T. J. Willmore, *An Introduction to Differential Geometry* (Clarendon Press, Oxford, England, 1959), Chap. 6, Sec. 2.

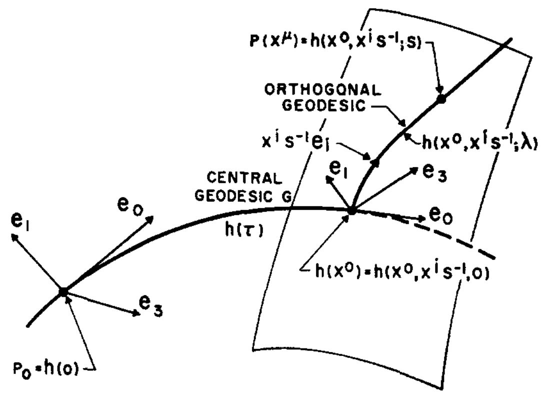

FIG. 1. Fermi normal coordinates are determined by a reference point P_0 and an orthonormal reference frame $\mathbf{e}_\mu$ there. The time axis G of the coordinates is the geodesic $h(\tau)$ tangent to $\mathbf{e}_0$ at P_0. The point $P(x^\mu)$, with given Fermi normal coordinates x^μ, is found by first following G for a proper time $\tau = x^0$, and then following a certain orthogonal geodesic at a proper distance $s = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{\frac{1}{2}}$. This second spacelike geodesic $h(x^0, x^i s^{-1}, \lambda)$, is chosen by requiring that for $\lambda = 0$, where it crosses G, its tangent has direction cosines $x^i s^{-1}$ relative to the base vectors $\mathbf{e}_i$ carried by parallel transport along G from P_0.

along G of the vectors $\mathbf{e}_i = \mathbf{e}_i(0)$ given at $P_0 = h(0)$.

The prescription for locating the point P whose Fermi normal coordinates are x^μ can now be given. We assume for simplicity that $\mathbf{e}_0$ is a timelike vector and $\mathbf{e}_i$ are spacelike. Then given x^μ, we construct at the point $h(x^0)$ [i.e., along G at $\tau = x^0$] the unit vector

$$\mathbf{v} = \alpha^i \mathbf{e}_i(x^0), \tag{6}$$

whose (spatial) direction cosines α^i are

$$\alpha^i = x^i/s, \tag{7}$$

with

$$s^2 = (x^i)^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{8}$$

There is then a unique geodesic

$$P = h(x^0; \alpha^i; \lambda), \tag{9}$$

with path parameter λ which starts at $h(x^0)$ and is tangent to $\mathbf{v}$ there. The point $P(x^\mu)$ with Fermi normal coordinates x^μ is found by proceeding along this geodesic a proper distance s, i.e.,

$$P(x^\mu) = h(x^0; \alpha^i; s). \tag{10}$$

This prescription is summarized in Fig. (1). When points P are represented by their coordinate values $y^{\mu'}(P)$ in some coordinate system[16] in which the

[16] Since we think of the metric or any other tensor as an object which is independent of our choice of coordinate system, we prefer that the indication of the particular coordinate system to which a set of tensor components $g_{\mu\nu}$ refers be placed on the component (index) part of the symbol rather than on the tensor part. Thus $g_{\mu\nu}$ and $g_{\mu'\nu'}$ are components of the same metric tensor in two coordinate systems, while, should the occasion arise, $g_{\mu\nu}$ and $g'_{\mu\nu}$ might represent two different metrics in a single coordinate system. See also the transformation laws of Eqs. (40) and (76).

metric components $g_{\mu'\nu'}(y^\alpha)$ are known, the constructions which we have prescribed lead to the relationship (10) with the point $P(x^\mu)$ given in terms of its coordinates $y^{\nu'}$ as $y^{\nu'}(x^\mu)$. These functions $y^{\nu'}(x^\mu)$ specify the coordinate transformation between the arbitrary coordinates $y^{\nu'}$ and the Fermi normal coordinates x^μ. We have not needed to mention these arbitrary initial coordinates $y^{\mu'}$ while prescribing the construction of the Fermi normal coordinates x^μ, and have avoided doing so to emphasize the fact that the point $P(x^\mu)$ corresponding to given values of x^μ is independent of the coordinate system $y^{\mu'}$ in which the computations may have been performed.

III. TANGENT VECTORS AND LIE BRACKETS

In the preceding construction of Fermi normal coordinates, the vectors which appeared were all used as tangents to curves. We want to recall here that *all* contravariant vectors can be thought of as tangents to curves and identified with the derivative with respect to the corresponding curve parameter.

Given a curve $y^\mu(t)$ in some coordinate system y^μ, the tangent vector

$$t^\mu = dy^\mu/dt \tag{11}$$

is clearly a contravariant vector, and can be used to compute derivatives $\partial/\partial t$ along the curve $y^\mu(t)$ by the rule

$$\frac{\partial f}{\partial t} \equiv \frac{df(y^\mu(t))}{dt} = \frac{dy^\mu}{dt}\frac{\partial f}{\partial y^\mu} \equiv t^\mu f_{,\mu}. \tag{12}$$

Conversely, given a contravariant vector field[17] $t^\mu(y^\alpha)$, we can solve the ordinary differential equations

$$dy^\mu/dt = t^\mu(y^\alpha(t)) \tag{11}$$

to obtain curves $y^\mu(t)$ with tangents t^μ.

The advantage of thinking of contravariant vectors t^μ as tangents to curves is that this helps us find a concrete mathematical object we can identify with the abstract vector **t** whose components t^μ appear in our computations. This object is the operation of differentiation along the curve whose tangent is **t**. That is, we write

$$\mathbf{t} \equiv \partial/\partial t. \tag{13}$$

The right-hand side of this identification is an operation which can be described in a *coordinate-independent way. The tangent,* **t** *or* $\partial/\partial t$, *to a curve* $P(t)$ *is the operation on scalar functions* $f(P)$ *defined by*

$$\partial f(P)/\partial t \equiv df(P(t))/dt, \tag{14}$$

i.e., by inserting the equation of the curve and taking an ordinary derivative. Since the operator $\partial/\partial t$ is applicable to all scalar functions, it can be applied in particular to those four scalar functions $y^\mu(P)$ we may be using as coordinates:

$$\partial y^\mu(P)/\partial t \equiv dy^\mu/dt = t^\mu. \tag{15}$$

In this way one can recover the components t^μ from the vector $\partial/\partial t$. Conversely, writing Eq. (12) in the form

$$\mathbf{t} \equiv \partial/\partial t = t^\mu\,\partial/\partial y^\mu, \tag{16}$$

we construct the contravariant vector $\partial/\partial t$ from a knowledge of its component t^μ. Equation (16) shows $\partial/\partial t$ as a linear combination, with coefficients t^μ, of four contravariant base vectors $\partial/\partial y^\mu$. These base vectors are tangents to the coordinate lines, e.g., $\partial/\partial y^0$ is the tangent $\partial/\partial t$ to the curve y^i = const, $y^0 = t$. We have frequent use for Eqs. (15) and (16) in what follows. In particular, Eq. (16) provides a method of displaying the components of a vector which simultaneously reminds us what coordinate system is being used and is, in this respect, superior to a statement of the form $\mathbf{t} = (t^0, t^1, t^2, t^3)$. We also find it convenient to be able to designate the components of a vector in several different ways, and thus write

$$(\partial/\partial t)^\mu = (\mathbf{t})^\mu = t^\mu = \partial y^\mu/\partial t. \tag{17}$$

Although we represent contravariant vectors **t** by the partial derivative symbol $\partial/\partial t$, it is not always possible to think of several vectors simultaneously as having the properties of standard partial derivatives.[18] In particular, consider the commutator of two tangent vector *fields* $\mathbf{u} \equiv \partial/\partial u$ and $\mathbf{v} \equiv \partial/\partial v$:

$$[\mathbf{u}, \mathbf{v}]f \equiv \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right). \tag{18}$$

Since **v** is a field, $\partial f/\partial v$ is a function and can be subsequently differentiated along a curve tangent to **u**. Thus the right-hand side of Eq. (18) is well defined, and evidently does not depend on the coordinates used to evaluate it. If we do pick a coordinate system, e.g., $\partial f/\partial v = v^\mu\,\partial f/\partial y^\mu$, Eq. (18)

[17] A vector t^μ given only at a point, or along a curve, etc., can always and in many ways be considered part of a vector field by arbitrarily defining $t^\mu(y^\alpha)$ at other points.

[18] A single vector $\partial/\partial t$ differs from a partial derivative by the possibility of vanishing; e.g., the tangent to a constant curve $P(t) = P_0$ is the zero vector $(\partial/\partial t) = 0$, since $\partial f/\partial t = df(P_0)/dt = 0$ for all functions f. However, in regions where $\partial/\partial t \neq 0$, coordinates can be introduced so that $\partial/\partial t = \partial/\partial y^0$ is a conventional partial derivative.

reads

$$[\mathbf{u}, \mathbf{v}]f = \left(u^\nu \frac{\partial v^\mu}{\partial y^\nu} - v^\nu \frac{\partial u^\mu}{\partial y^\nu}\right) \frac{\partial f}{\partial y^\mu}, \tag{19}$$

or

$$[\mathbf{u}, \mathbf{v}] = (v^\mu{}_{,\nu} u^\nu - u^\mu{}_{,\nu} v^\nu) \frac{\partial}{\partial y^\mu}. \tag{20}$$

As a linear combination of the base vectors $\partial/\partial y^\mu$, the object $[\mathbf{u}, \mathbf{v}]$ evidently is itself a contravariant vector, called the *Lie Bracket* of $\mathbf{u}$ and $\mathbf{v}$. Its components are displayed in Eq. (20). In case (as in this paper) a covariant derivative is defined, Eq. (20) can be rewritten as

$$[\mathbf{u}, \mathbf{v}]^\mu = v^\mu{}_{;\nu} u^\nu - u^\mu{}_{;\nu} v^\nu, \tag{21}$$

since the symmetry of the $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$ lets them cancel here in any case.

We most often wish to use Eq. (21) in the case where we know $[\mathbf{u}, \mathbf{v}] = 0$. This is true whenever $\mathbf{u}$ and $\mathbf{v}$ can be thought of as tangents to coordinate lines in a surface. That is, let $P(u, v)$ be the equation of a "surface" parameterized by u, v, and let $\mathbf{u} = \partial/\partial u$ be the tangent to lines of constant v in this surface, and similarly $\mathbf{v} = \partial/\partial v$ is tangent to lines of constant u. Then $[\mathbf{u}, \mathbf{v}]$ can be evaluated from Eq. (18) by setting $f = f(P(u, v))$ on the right-hand side. The derivatives are then standard partial derivatives which commute, so $[\mathbf{u}, \mathbf{v}] = 0$.[19] [This derivation requires only that $P(u, v)$ be a differentiable point-valued function; it actually represents a *two*-dimensional surface only if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors.]

IV. REGULARITY OF FERMI NORMAL COORDINATES

According to the construction of Sec. II, Fermi normal coordinates are specified in terms of the solution

$$P = h(\tau; \alpha^i; \lambda) \tag{22}$$

of the geodesic equation describing a geodesic which begins ($\lambda = 0$) at the point

$$h(\tau; \alpha^i; 0) = h(\tau) \tag{23a}$$

on the central geodesic G, and whose tangent there is

$$(\partial/\partial\lambda)_{\lambda=0} = \alpha^i \mathbf{e}_i(\tau). \tag{23b}$$

As used in Sec. II, the parameters α^i satisfied $(\alpha^i)^2 = 1$, but we ignore this condition now and consider all values of the α^i. We first prove that

$$h(\tau; s\alpha^i; \lambda) = h(\tau; \alpha^i; s\lambda) \tag{24}$$

holds for all s (by rescaling the path parameter λ) so that Eq. (10) defining Fermi normal coordinates may be replaced by

$$P(x^\mu) = h(x^0; x^i; 1). \tag{25}$$

This form allows us to verify more easily the differentiability of the inverse relationship, $x^\mu(P)$, i.e., of the coordinate functions.

We regard $h(\tau)$ and $h(\tau; \alpha^i; \lambda)$ as the point-valued function of one and five real variables, respectively, computed without regard to any interpretation placed on their real-number arguments. (In contrast, common usage for real-valued functions dictates that $f(x^\alpha)$ and $f(y^{\alpha'})$ mean different functions of their four real arguments so as to represent the same function of points $f(P)$ in two different coordinate systems.) Then, to prove Eq. (24) we rewrite Eq. (22) in some arbitrary regular coordinate system $y^{\mu'}(P)$ as

$$y^{\mu'} = h^{\mu'}(\tau; \alpha^i; \lambda). \tag{26}$$

The functions $h^{\mu'}$ are simply the *unique* solutions of the differential equations

$$\frac{d^2h^{\mu'}}{d\lambda^2} + [\Gamma_{\alpha'}{}^{\mu'}{}_{\beta'}]_{y'=h'} \frac{dh^{\alpha'}}{d\lambda}\frac{dh^{\beta'}}{d\lambda} = 0, \tag{27}$$

which satisfy the initial conditions

$$h^{\mu'}(\tau; \alpha^i; 0) = h^{\mu'}(\tau), \tag{28}$$

and

$$\frac{dh^{\mu'}(\tau; \alpha^i; 0)}{d\lambda} \equiv \left(\frac{\partial}{\partial\lambda}\right)^{\mu'}_{\lambda=0} = \alpha^i(\mathbf{e}_i(\tau))^{\mu'}, \tag{29}$$

where $y^{\mu'} = h^{\mu'}(\tau)$ is the central geodesic G. After remarking that the differential equation (27) is unchanged upon replacing λ by $s\lambda$, we prove Eq. (24) by verifying that, as function of λ, $h^{\mu'}(\tau; s\alpha^i; \lambda)$ and $h^{\mu'}(\tau; \alpha^i; s\lambda)$ not only satisfy the same differential equation (27), but also the same initial condition. For, each reduces to $h^{\mu'}(\tau)$ for $\lambda = 0$ and has a first derivative $s\alpha^i(\mathbf{e}_i(\tau))^{\mu'}$ at $\lambda = 0$. Thus, by the Uniqueness Theorem[20] for solutions of differential equations we have

$$h^{\mu'}(\tau; s\alpha^i; \lambda) = h^{\mu'}(\tau; \alpha^i; s\lambda), \tag{24a}$$

[19] To see that the Lie Bracket does not *always* vanish, an example suffices. For the unit vectors $\mathbf{e}_\theta = \partial/\partial\theta$ and $\mathbf{e}_\varphi = (\sin\theta)^{-1}\partial/\partial\varphi$ on the unit sphere, compute from Eq. (20)

$$[\mathbf{e}_\theta, \mathbf{e}_\varphi] = -\cot\theta\,\mathbf{e}_\varphi \neq 0.$$

[20] See, for example, F. J. Murray and K. S. Miller, *Existence Theorems*, (New York University Press, New York, 1954), Chap. 2, Theorems 1, 3; Chap. 3, Theorem 2; Chap. 5, Theorem 6. A discussion of the properties of geodesics from which we have borrowed much is found in H. Seifert and W. Threlfall, *Variationsrechnung im Grossen* (B. G. Teubner, Leipzig, 1938), footnote 20, p. 97.

which represents Eq. (24) in the $y^{\mu'}$ coordinate system.

The definition of Fermi normal coordinates in Eq. (25) gives now the transformation law

$$y^{\mu'}(x^\alpha) = h^{\mu'}(x^0; x^i; 1). \tag{30}$$

By a standard theorem,[20] the solutions of ordinary differential equations are differentiable functions of the initial conditions, so we have established the differentiability of $y^{\mu'}(x^\alpha)$. To show the existence of a differentiable inverse relation $x^\mu(y^\alpha)$, representing the coordinate functions $x^\mu(P)$, we must show that the Jacobian $|\partial y^{\mu'}/\partial x^\alpha|$ does not vanish.[20]

The condition of a nonvanishing Jacobian is precisely the condition that the coordinate axes do not collapse, i.e., that the vectors $\partial/\partial x^\mu$ be linearly independent. For, when we form the components of, say, $\partial/\partial x^0$ in the $y^{\mu'}$ frame, they are $\partial y^{\mu'}/\partial x^0$, so that the determinant formed from the components of the four vectors $\partial/\partial x^\alpha$ is

$$J \equiv \det\,(\partial y^{\mu'}/\partial x^\alpha), \tag{31}$$

and $J \neq 0$ is equivalent to the linear independence of these vectors. We prove $J \neq 0$ by showing that, along the central geodesic G,

$$(\partial/\partial x^\mu)_G = \mathbf{e}_\mu(\tau). \tag{32}$$

Then, since $\mathbf{e}_\mu(\tau)$ are orthonormal vectors, they are linearly independent and $J \neq 0$ on G. By continuity, then, we have $J \neq 0$ in some neighborhood of G.

The basic fact we need in order to prove the equation $(\partial/\partial x^\mu)_G = \mathbf{e}_\mu(\tau)$ in the preceding argument is the description in Fermi normal coordinates of the geodesics entering their construction. This is also the basis from which we will compute all other properties of Fermi normal coordinates. Consider then the curve $P(\lambda)$ defined in Fermi normal coordinates by

$$\begin{aligned} x^0 &= \tau = \text{const}, \\ x^i &= \alpha^i\lambda, \qquad \alpha^i = \text{const}. \end{aligned} \tag{33}$$

According to Eqs. (25) and (24), this curve is given by

$$P(\lambda) = h(\tau; \alpha^i\lambda; 1) = h(\tau; \alpha^i; \lambda), \tag{34}$$

and is, therefore, that geodesic whose tangent $\partial/\partial\lambda$ is given by Eq. (23b) at the point $x^i = 0$, $x^0 = \tau$ corresponding to $\lambda = 0$. But the components of $\partial/\partial\lambda$ can be computed from Eqs. (33), and are $(\partial/\partial\lambda)^0 = 0$ and $(\partial/\partial\lambda)^i = \alpha^i$, so

$$\partial/\partial\lambda = \alpha^i\,\partial/\partial x^i. \tag{35}$$

Comparing this with Eq. (23b) gives

$$(\partial/\partial x^i)_{x^i=0} = \mathbf{e}_i(x^0), \tag{32a}$$

since the α^i are arbitrary. Similarly, from Eqs. (25), (24), and (23a) we see that the curve $P(\tau)$ defined by

$$\begin{aligned} x^0 &= \tau, \\ x^i &= 0, \end{aligned} \tag{36}$$

is given by

$$P(\tau) = h(\tau; 0; 1) = h(\tau; 0; 0) = h(\tau), \tag{36b}$$

and is the central geodesic G whose tangent $\partial/\partial\tau$ is $\mathbf{e}_0(\tau)$. But the components of the tangent $(\partial/\partial\tau)$ are easily computed from Eq. (36) and give

$$(\partial/\partial x^0)_{x^i=0} = \mathbf{e}_0(x^0). \tag{32b}$$

To recapitulate, the question of the Jacobian or of the linear independence of the $\partial/\partial x^\mu$, reduces by Eq. (32) to the linear independence of the $\mathbf{e}_\mu(\tau)$. But $\mathbf{e}_\mu(\tau)$ are orthonormal, since they are defined by parallel displacement of the orthonormal vectors $\mathbf{e}_\mu(0)$, and parallel displacement preserves inner products,

$$\mathbf{e}_\mu(\tau)\cdot\mathbf{e}_\nu(\tau) = \eta\mu\nu. \tag{37}$$

V. THE FERMI CONDITIONS

We have actually already proven that in Fermi normal coordinates the metric is rectangular on G. For by definition, the metric components are the matrix of inner products of the base vectors, i.e.,

$$g_{\mu\nu}(x^\alpha) = (\partial/\partial x^\mu)\cdot(\partial/\partial x^\nu), \tag{38}$$

so Eqs. (32) and (37) give

$$g_{\mu\nu}(x^0; 0) \equiv g_{\mu\nu}|_G = \eta_{\mu\nu}. \tag{39}$$

It may, nevertheless, be instructive to see this equation arise by applying the tensor transformation law to the metric components $g_{\alpha'\beta'}(y^{\sigma'})$ of some original coordinate system $y^{\sigma'}$:

$$g_{\mu\nu}|_G = g_{\alpha'\beta'}\left(\frac{\partial y^{\alpha'}}{\partial x^\mu}\right)_G\left(\frac{\partial y^{\beta'}}{\partial x^\nu}\right)_G = \mathbf{e}_\mu\cdot\mathbf{e}_\nu = \eta_{\mu\nu}. \tag{40}$$

In the central equality here we recalled that $(\partial y^{\alpha'}/\partial x^\mu)_G$ are the components of $(\partial/\partial x^\mu)_G = \mathbf{e}_\mu(x^0)$ in the $y^{\alpha'}$ frame.

In order to show that $\Gamma_\mu{}^\alpha{}_\nu\,|_G = 0$ holds in Fermi normal coordinates, we begin by considering the consequences of the fact that the curve $x^0 = \tau$, $x^i = \alpha^i\lambda$, satisfies the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_\alpha{}^\mu{}_\beta\,\frac{dx^\alpha}{d\lambda}\,\frac{dx^\beta}{d\lambda} = 0. \tag{41}$$

Substitution gives

$$\Gamma_i{}^\mu{}_j(x^0 = \tau; x^i = \alpha^i\lambda)\alpha^i\alpha^j = 0, \tag{42a}$$

which holds in particular for $\lambda = 0$:

$$\Gamma_i{}^\mu{}_j(x^0 = \tau; x^i = 0)\alpha^i\alpha^j = 0. \tag{42b}$$

But now the $\Gamma_{ij}{}^\mu$ are independent of α^i and symmetric in i and j, while the α^i are arbitrary, so

$$\Gamma_i{}^\mu{}_j|_G = 0. \tag{43}$$

To show that the other Christoffel symbols vanish, we recall that each of the vectors $\mathbf{e}_\mu(\tau)$ satisfies the equation of parallel displacement along the central geodesic $x^0 = \tau$, $x^i = 0$. In general, to parallel displace a vector $\mathbf{u}$ along a curve $P(t)$, one solves the differential equation (linear)

$$u^\mu{}_{;\nu}t^\nu = du^\mu/dt + u^\alpha\Gamma_\alpha{}^\mu{}_\beta t^\beta. \tag{44}$$

We may take $\mathbf{u}$ to be any of the vectors $\mathbf{e}_\sigma(\tau) = (\partial/\partial x^\sigma)_G$ whose components in Fermi normal coordinates are therefore $(\mathbf{e}_\sigma)^\mu = \delta_\sigma{}^\mu$, and $\mathbf{t}$ becomes $\partial/\partial\tau = \mathbf{e}_0 = \delta_0{}^\beta(\partial/\partial x^\beta)_G$. Thus, from Eq. (44) we obtain

$$\Gamma_\sigma{}^\mu{}_0|_G = 0. \tag{45}$$

Combining Eqs. (43) and (45) gives the second of the Fermi conditions,

$$\Gamma_\alpha{}^\mu{}_\beta|_G = 0. \tag{46}$$

Since this implies $(\partial g_{\alpha\beta}/\partial x^\mu)_G = 0$, we have evaluated the first two terms in a Taylor expansion of the Fermi normal metric. The quadratic terms, which require us to evaluate $(\partial^2 g_{\alpha\beta}/\partial x^i\partial x^j)_G$, is computed in Sec. VII, after a diversion to review the computational technique we will use.

VI. EQUATION OF GEODESIC DEVIATION[7,21]

The construction of Fermi normal coordinates involves families of geodesics. Let us consider only a one-parameter family of geodesics for the present, say $P(n, s)$, where for each fixed value at $n = n_0$, $P(n_0, s)$ satisfies the geodesic differential equation with s as path parameter. The tangent vector $\mathbf{s} = \partial/\partial s$ can then be thought of as the generator of infinitesimal translations along geodesic n, while $\mathbf{n} = \partial/\partial n$ is the generator of infinitesimal translations along a curve $P(n, s_0)$ connecting corresponding points (same value of s) on adjacent geodesics. Along a fixed geodesic, $\mathbf{n}$ cannot vary arbitrarily, since the adjacent geodesic can be determined by only two points lying on it. These restrictions on $\mathbf{n}$ are expressed by the equation of geodesic deviation which is a differential equation satisfied by $\mathbf{n}$ along each geodesic, i.e., as a function of s.

To derive the equation of geodesic deviation, we begin with the geodesic equation in the form[9]

$$\delta\mathbf{s}/\delta s = 0, \tag{47}$$

where $\delta/\delta s$ is the covariant derivative along s. This is, of course, just an abbreviation for

$$s^\mu{}_{;\nu}s^\nu \equiv ds^\mu/ds + s^\alpha\Gamma_\alpha{}^\mu{}_\beta s^\beta = 0, \tag{48}$$

but the more compact notation lets us outline the derivation without computations. Since Eq. (47) holds for all values of n, we may differentiate it to obtain

$$\frac{\delta}{\delta n}\left(\frac{\delta\mathbf{s}}{\delta s}\right) = 0. \tag{49}$$

As the difference of two geodesic equations, this should be an equation for the difference vector $\mathbf{n}$, that is, $\mathbf{n}$ should appear differentiated, rather than as a derivative. The relationship which achieves this is Eq. (21) which can be written

$$0 = [\mathbf{n}, \mathbf{s}] \equiv \delta\mathbf{s}/\delta n - \delta\mathbf{n}/\delta s. \tag{50}$$

[The Lie bracket $[\mathbf{n}, \mathbf{s}]$ vanishes since n and s parameterize the surface $P(n, s)$.] Before this relation can be employed in Eq. (49), however, the covariant derivatives must be written in the opposite order,

$$0 = \frac{\delta}{\delta n}\left(\frac{\delta\mathbf{s}}{\delta s}\right) = \frac{\delta}{\delta s}\left(\frac{\delta\mathbf{s}}{\delta n}\right) + \left\{\frac{\delta}{\delta n}\frac{\delta}{\delta s} - \frac{\delta}{\delta s}\frac{\delta}{\delta n}\right\}\mathbf{s}, \tag{51}$$

or

$$\delta^2\mathbf{n}/\delta s^2 + [\delta/\delta n, \delta/\delta s]\mathbf{s} = 0. \tag{52}$$

When the commutator[22] of covariant derivatives here is expressed in terms of the curvature tensor, this Eq. (52) is the equation of geodesic deviation. The computation is

$$\begin{aligned}\{[\delta/\delta n, \delta/\delta s]\mathbf{s}\}^\mu &= (s^\mu{}_{;\alpha}s^\alpha)_{;\beta}n^\beta - (s^\mu{}_{;\beta}n^\beta)_{;\alpha}s^\alpha \\ &= (s^\mu{}_{;\alpha\beta} - s^\mu{}_{;\beta\alpha})s^\alpha s^\beta + s^\mu{}_{;\nu}(s^\nu{}_{;\sigma}n^\sigma - n^\nu{}_{;\sigma}s^\sigma) \\ &= s^\sigma R_\sigma{}^\mu{}_{\beta\alpha}n^\beta s^\alpha + s^\mu{}_{;\nu}[\mathbf{n}, \mathbf{s}]^\nu \\ &= s^\alpha R_\alpha{}^\mu{}_{\nu\beta}s^\beta n^\nu,\end{aligned} \tag{53}$$

[21] For physical applications see F. A. E. Pirani, Acta Phys. Polon. **15**, 389 (1956); Phys. Rev. **105**, 1089 (1957); and J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961), Chap. 8.

[22] The entire derivation may be regarded as a process of evaluating the commutators which relate Eq. (49) in the curious form

$$\frac{\delta}{\delta n}\frac{\delta}{\delta s}\left(\frac{\partial}{\partial s}\right) = 0$$

to an equation whose leading term is

$$\frac{\delta^2 n}{\delta s^2} = \frac{\delta}{\delta s}\frac{\delta}{\delta s}\left(\frac{\partial}{\partial n}\right).$$

where we again use $[\mathbf{n}, \mathbf{s}] = 0$. Thus, we find for the equation of geodesic deviation

$$\delta^2 n^\mu/\delta s^2 + (s^\alpha R_\alpha{}^\mu{}_{\nu\beta} s^\beta) n^\nu = 0. \tag{54}$$

Let us briefly outline the principal properties of this equation. It is a second-order, *linear*, ordinary differential equation for $\mathbf{n}$ as a function of s. In addition to the trivial solution $\mathbf{n} = 0$, it also obviously has the solution $\mathbf{n} = \mathbf{s}$. A perhaps not so obvious solution which can, however, be easily verified is $\mathbf{n} = s\mathbf{s}$. The solution $\mathbf{n} = (A's + B')\mathbf{s}$ (where the adjacent geodesics coincide with the original one but are parameterized differently) clearly satisfies

$$\mathbf{n}\cdot\mathbf{s} = As + B, \tag{55}$$

where A and B are constants $[A = A'(\mathbf{s}\cdot\mathbf{s})$ while $\mathbf{s}\cdot\mathbf{s} = \text{const}$ according to the geodesic equations]. We can further show that *every* solution $\mathbf{n}$ satisfies Eq. (55) by using the product rule of covariant differentiation and the geodesic equation

$\delta s_\mu/\delta s = 0$ to compute

$$\frac{\partial^2}{\partial s^2}(s_\mu n^\mu) = \frac{\delta^2}{\delta s^2}(s_\mu n^\mu) = \frac{\delta}{\delta s}\left(s_\mu \frac{\delta n^\mu}{\delta s}\right) = s_\mu \frac{\delta^2 n^\mu}{\delta s^2}$$
$$= -s^\alpha s^\mu R_{\alpha\mu\nu\beta} n^\nu s^\beta = 0. \tag{56}$$

The constant A in Eq. (55) is related to the normalization of the geodesic parameter, for by using the Lie bracket relation $[\mathbf{n}, \mathbf{s}] = 0$, we find

$$A = \frac{\delta}{\delta s}(s_\mu n^\mu) = s_\mu \frac{\delta n^\mu}{\delta s} = s_\mu \frac{\delta s^\mu}{\delta n} = \frac{1}{2}\frac{\partial}{\partial n}(s_\mu s^\mu). \tag{57}$$

Unless $\mathbf{s}\cdot\mathbf{s} = 0$, we can always modify any solution $\mathbf{n}$ of the geodesic deviation equation, adding terms of the form $(As + B)\mathbf{s}$, to obtain a solution satisfying $\mathbf{n}\cdot\mathbf{s} = 0$. This modification corresponds to a linear change in the parameterization of the adjacent geodesic, which is of course consistent with the geodesic equation. According to Eq. (57) the condition $\mathbf{n}\cdot\mathbf{s} = 0$ is consistent with the standard normalization $\mathbf{s}\cdot\mathbf{s} = \pm 1$ for geodesic parameters.

VII. QUADRATIC TERMS IN THE FERMI METRIC

A power-series expansion of the metric in Fermi normal coordinates is determined by the derivatives $g_{\mu\nu,ij\cdots k}|_G$. The linear terms $g_{\mu\nu,i}|_G\, x^i$ were shown to vanish in Sec. V, where we found that on the central geodesic G all the Christoffel symbols vanish. In this section we will compute the quadratic terms, $\frac{1}{2}g_{\mu\nu,ij}|_G\, x^i x^j$, by first computing $\Gamma_\mu{}^\alpha{}_{\nu,\beta}|_G$.

Since the equation $\Gamma_\mu{}^\alpha{}_\nu = 0$ holds for all x^0 at $x = 0$, it may be differentiated with respect to x^0 to give

$$\Gamma_\mu{}^\alpha{}_{\nu,0}|_G = 0. \tag{58a}$$

Also using $\Gamma_\mu{}^\alpha{}_\nu|_G = 0$, we note that on G the definition of the Riemann tensor[6] reduces to two terms, and, in particular, from Eq. (58a) we find

$$\Gamma_\mu{}^\alpha{}_{0,\nu}|_G = R_\mu{}^\alpha{}_{\nu 0}|_G. \tag{58b}$$

The remaining derivatives of the affine connection are

$$\Gamma_i{}^\mu{}_{j,k}|_G = -\tfrac{1}{3}(R_i{}^\mu{}_{jk} + R_j{}^\mu{}_{ik})|_G, \tag{58c}$$

as we now show by use of the equation of geodesic deviation. Note that this last equation implies a symmetry

$$\Gamma_i{}^\mu{}_{j,k}|_G + \Gamma_j{}^\mu{}_{k,i}|_G + \Gamma_k{}^\mu{}_{i,j}|_G = 0, \tag{58d}$$

peculiar to these coordinates which follows from the corresponding "triple symmetry" of the Riemann tensor.

The family of geodesics $P(\lambda) = h(\tau; \alpha^i; \lambda)$ used in constructing Fermi normal coordinates provides us with four vectors, $\partial/\partial\tau$ and $\partial/\partial\alpha^i$, which (since they generate displacements between adjacent geodesics) must each satisfy the equation of geodesic deviation as functions of λ for fixed τ, α^i [In contrast, $P(\tau) = h(\tau; \alpha^i; \lambda)$ is not a geodesic unless $\lambda\alpha^i = 0$, so we have no *family* of geodesics with tangents $\partial/\partial\tau$, and neither $\partial/\partial\alpha^i$ nor $\partial/\partial\lambda$ satisfies the equation of geodesic deviation as a function of τ, even for $\lambda\alpha^i = 0$.] Although there are moderate amounts of computation involved in what follows now, the basic idea is quite simple. In the geodesic deviation equation ("$\delta^2\mathbf{n} + R\mathbf{n} = 0$") we insert known solutions, $\mathbf{n} = \partial/\partial\tau$ or $\partial/\partial\alpha^i$. At the point $\lambda = 0$ (i.e., on G where $\Gamma_\mu{}^\alpha{}_\nu = 0$), the second covariant derivative term will reduce to the derivative of a Christoffel symbol evaluated on G, and the only other term in the equation will be the curvature term, so we will obtain a formula "$\partial\Gamma = R$", i.e., Eqs. (58).

The family of geodesics $P = h(\tau; \alpha^i; \lambda)$ is described in Fermi normal coordinates by the equations

$$x^0 = \tau, \qquad x^i = \alpha^i\lambda. \tag{59}$$

The components of the deviation vectors $\mathbf{n}$, computed using Eq. (17), are then

$$\partial/\partial\tau = \partial/\partial x^0 \equiv \delta_0{}^\mu\, \partial/\partial x^\mu, \tag{60}$$

$$\partial/\partial\alpha^i = \lambda\, \partial/\partial x^i \equiv \lambda\delta_i{}^\mu\, \partial/\partial x^\mu. \tag{61}$$

Similarly, the vector $\mathbf{s}$ tangent to the geodesic is

$$\partial/\partial\lambda = \alpha^i\, \partial/\partial x^i. \tag{62}$$

We use the components displayed here in the geodesic deviation equation (54) which in detail becomes

$$\frac{d^2n^\mu}{ds^2} + 2\frac{dn^\sigma}{ds}\Gamma_\sigma{}^\mu{}_\alpha s^\alpha + n^\sigma s^\alpha s^\beta R_\alpha{}^\mu{}_{\sigma\beta}$$
$$+ n^\sigma s^\alpha s^\beta(\Gamma_\sigma{}^\mu{}_{\alpha,\beta} + \Gamma_\sigma{}^\tau{}_\alpha\Gamma_\tau{}^\mu{}_\beta - \Gamma_\sigma{}^\mu{}_\tau\Gamma_\alpha{}^\tau{}_\beta) = 0. \quad (63)$$

The case $\mathbf{n} = \partial/\partial\tau$ merely leads to some of Eqs. (58b) again, so we treat only the cases $\mathbf{n} = \partial/\partial\alpha^i$. Then dn^σ/ds becomes $d(\lambda\delta_i{}^\sigma)/d\lambda = \delta_i{}^\sigma$, and

$$d^2n^\mu/d\lambda^2 = 0. \quad (63a)$$

But, since $n^\mu = \lambda\delta_i{}^\mu$ vanishes for $\lambda = 0$, Eq. (63) is trivial on G unless we divide through by λ before setting $\lambda = 0$. In order to accomplish this, the second term in Eq. (63) can be expanded in powers of λ;

$$2\delta_i{}^\sigma\Gamma_\sigma{}^\mu{}_j\alpha^j = 2\Gamma_i{}^\mu{}_j|_G\,\alpha^j + 2\lambda\left(\frac{\partial}{\partial\lambda}\Gamma_i{}^\mu{}_j\alpha^j\right)\Big|_G$$
$$= 2\lambda\Gamma_i{}^\mu{}_{j,k}|_G\,\alpha^j\alpha^k + O(\lambda^2). \quad (64a)$$

Then, at $\lambda = 0$ where $\Gamma_\mu{}^\alpha{}_\nu = 0$, we obtain from Eq. (63)

$$(3\Gamma_i{}^\mu{}_{j,k} + R_j{}^\mu{}_{ik})|_G\,\alpha^j\alpha^k = 0, \quad (64b)$$

or

$$(\Gamma_i{}^\mu{}_{j,k} + \Gamma_i{}^\mu{}_{k,j})|_G = -\tfrac{1}{3}(R_j{}^\mu{}_{ik} + R_k{}^\mu{}_{ij})|_G. \quad (64c)$$

This equation can be solved for $\Gamma_i{}^\mu{}_{j,k}|_G$ by adding to it one cyclic permutation,

$$(\Gamma_j{}^\mu{}_{k,i} + \Gamma_j{}^\mu{}_{i,k})|_G = -\tfrac{1}{3}(R_k{}^\mu{}_{ji} + R_i{}^\mu{}_{jk})|_G, \quad (64d)$$

and subtracting another,

$$(\Gamma_k{}^\mu{}_{i,j} + \Gamma_k{}^\mu{}_{j,i})|_G = -\tfrac{1}{3}(R_i{}^\mu{}_{kj} + R_j{}^\mu{}_{ik})|_G. \quad (64e)$$

The result, after using the symmetry of the connection $\Gamma_\mu{}^\alpha{}_\nu = \Gamma_\nu{}^\alpha{}_\mu$, is just Eq. (58c).

From the definition of Christoffel symbols,

$$g_{\mu\nu,\alpha} = g_{\mu\sigma}\Gamma_\nu{}^\sigma{}_\alpha + g_{\sigma\nu}\Gamma_\mu{}^\sigma{}_\alpha, \quad (64f)$$

we find by differentiation that

$$g_{\mu\nu,\alpha\beta}|_G = \eta_{\mu\sigma}\Gamma_\nu{}^\sigma{}_{\alpha,\beta}|_G + \eta_{\sigma\nu}\Gamma_\mu{}^\sigma{}_{\alpha,\beta}|_G. \quad (64g)$$

Thus, Eqs. (58) imply that

$$g_{\mu\nu,0\alpha}|_G = 0, \quad (64h)$$

and that, for $g_{\mu\nu,ij}|_G$, we have

$$g_{00,ij}|_G = 2R_{0i0j}|_G, \quad (65a)$$
$$g_{0k,ij}|_G = \tfrac{2}{3}(R_{0jki} + R_{0ikj})|_G, \quad (65b)$$
$$g_{lm,ij}|_G = \tfrac{1}{3}(R_{iljm} + R_{imjl})|_G. \quad (65c)$$

To summarize all the information we have obtained about the metric in Fermi normal coordinates, we can write the Taylor series

$$g_{00} = -1 + R_{0l0m}|_G\,x^lx^m + \cdots, \quad (66a)$$
$$g_{0i} = 0 \quad + \tfrac{2}{3}R_{0lim}|_G\,x^lx^m + \cdots, \quad (66b)$$
$$g_{ij} = \delta_{ij} + \tfrac{1}{3}R_{iljm}|_G\,x^lx^m + \cdots. \quad (66c)$$

Here the dependence of the metric on the spatial coordinates x^i is shown explicitly, while its dependence on x^0 is contained entirely in the curvature components which are evaluated at $x^i = 0$ for each x^0.

VIII. AN EXAMPLE

We compute the Schwarzschild metric to quadratic order, as in Eq. (66), in Fermi normal coordinates surrounding a radial geodesic. In Schwarzschild coordinates, which we will call $y^{\mu'}$ or T, R, Θ, Φ, the metric components $g_{\mu'\nu'}$ are displayed in the form

$$ds^2 = g_{\mu'\nu'}\,dy^{\mu'}\,dy^{\nu'}$$
$$= -X\,dT^2 + X^{-1}\,dR^2 + R^2\,d\Theta^2 + R^2\sin^2\Theta\,d\Phi^2, \quad (67)$$

where

$$X = 1 - 2M/R. \quad (68)$$

To find the equations of a radial geodesic, $T(t)$, $R(t)$, with Θ and Φ constant, one may replace the geodesic equations by two first integrals; one is the normalization of proper time

$$1 = XT'^2 - X^{-1}R'^2, \quad (69)$$

and the other is a dimensionless energy parameter

$$k = XT', \quad (70)$$

which yields $R(t)$ by quadratures, and expresses k of the metric. (The primes here indicate derivatives with respect to *proper* time t along this geodesic.) Eliminating T' gives

$$k^2 = X + R'^2 = 1 - 2M/R_0, \quad (71)$$

which yields $R(t)$ by quadratures, and expresses k in terms of the maximum radius R_0 along the path, where $R' = 0$. The integration gives a cycloid

$$R = \tfrac{1}{2}R_0(1 + \cos\omega),$$
$$t = \tfrac{1}{2}R_0(R_0/2M)^{\frac{1}{2}}(\omega + \sin\omega). \quad (72)$$

Either R or the cycloid parameter ω can be used in place of proper time t to identify points on this geodesic, and thus serve as a time coordinate in the comoving frame. Thus,

$$dt^2 = \frac{dR^2}{2M/R - 2M/R_0} = \frac{R_0}{2M}R^2\,d\omega^2. \quad (73)$$

After choosing a geodesic, the next step in constructing Fermi normal coordinates is to choose an orthonormal frame along the geodesic. The timelike base vector must be the tangent $\partial/\partial t$, and the symmetry of the present example determines the others. Thus,

$$\begin{aligned} \mathbf{e}_0 &= \partial/\partial t|_G = T'\,\partial/\partial T + R'\,\partial/\partial R, \\ \mathbf{e}_1 &= \partial/\partial x|_G = X^{-1}R'\,\partial/\partial T + XT'\,\partial/\partial R, \\ \mathbf{e}_2 &= \partial/\partial y|_G = 1/R\,\partial/\partial\Theta, \\ \mathbf{e}_3 &= \partial/\partial z|_G = 1/R\sin\Theta\,\partial/\partial\Phi, \end{aligned} \tag{74}$$

where x^μ or $xyzt$ are to be Fermi normal coordinates. It is also easily verified from the components $(\mathbf{e}_\alpha)^{\mu'}$ displayed here that these vectors satisfy the necessary parallel transport condition

$$\delta\mathbf{e}_\alpha/\delta t = 0 = (\mathbf{e}_\alpha)^{\mu'}{}_{;\nu'}(\mathbf{e}_0)^{\nu'}. \tag{75}$$

We must now compute the curvatures in the Fermi frame by the tensor transformation law

$$R_{\alpha\beta\gamma\delta} = R_{\mu'\nu'\sigma'\tau'}(\mathbf{e}_\alpha)^{\mu'}(\mathbf{e}_\beta)^{\nu'}(\mathbf{e}_\gamma)^{\sigma'}(\mathbf{e}_\delta)^{\tau'}, \tag{76}$$

which states that a tensor component is the contraction of the tensor with the base vectors indicated by the indices. The Fermi base vectors $\mathbf{e}_\alpha$ we have in Eq. (74), while the curvature components $R_{\mu'\nu'\sigma'\tau'}$ with respect to the Schwarzschild frame are well known as

$$\begin{aligned} R_{1'0'1'0'} &= 2M/R^3, \\ R_{3'0'3'0'} &= -(MX/R)\sin^2\Theta, \\ R_{1'2'1'2'} &= M/RX, \\ R_{2'0'2'0'} &= -MX/R, \\ R_{2'3'2'3'} &= -2MR\sin^2\Theta, \\ R_{1'3'1'3'} &= (M/RX)\sin^2\Theta. \end{aligned} \tag{77}$$

(Here and below, only the independent nonvanishing components are listed.) The computation then yields

$$\begin{aligned} R_{1010} &= 2M/R^3, \\ R_{2020} &= R_{3030} = -M/R^3, \\ R_{1212} &= R_{1313} = M/R^3, \\ R_{2323} &= -2M/R^3. \end{aligned} \tag{78}$$

Some of the simplicity of Eq. (78) as compared to Eq. (77) was, of course, to be expected, for the Fermi frame is orthonormal so that all components must at least have the same dimensions, and the equivalence of the Θ and Φ directions must become evident. However, a very surprising feature is that the gravitational field gradients in Eq. (78) depend on the observer's position R, but *not* upon his velocity R' (or energy k) with respect to the mass M. Thus the preferred rest frame indicated locally by the Killing vector field $\partial/\partial T$ cannot be recognized by an observer who measures all the gravitational field gradients (78) at one point. He can only discover the direction of the vector $\partial/\partial T$ by finding a velocity (i.e. direction in the $R - T$ plane) which makes the field gradients constant in time, i.e., by measuring $R_{\mu\nu\sigma\tau;\alpha}$.

The Fermi normal metric from Eq. (66) is

$$\begin{aligned} ds^2 = &-\left[1 + \frac{M}{R^3}(y^2 + z^2 - 2x^2)\right]dt^2 \\ &- \frac{2M}{3R^3}[xz\,dx\,dz + xy\,dx\,dy - 2yz\,dy\,dz] \\ &+ \left[1 + \frac{M}{3R^3}(y^2 + z^2)\right]dx^2 \\ &+ \left[1 + \frac{M}{3R^3}(x^2 - 2z^2)\right]dy^2 \\ &+ \left[1 + \frac{M}{3R^3}(x^2 - 2y^2)\right]dz^2. \end{aligned} \tag{79}$$

The entire dependence of this metric on t is through the geodesic equation (72) which gives $R(t)$.

A more compact form for the Fermi metric (79) is obtained by introducing spherical coordinates r, θ, φ related to x, y, z by the standard formulas. Taking the x direction as the polar axis we get a diagonal metric,

$$\begin{aligned} ds^2 = &-(1 - q\mu)\,dt^2 + dr^2 + (1 + \tfrac{1}{3}\mu)(r\,d\theta)^2 \\ &+ (1 + \tfrac{1}{3}q\mu - \tfrac{1}{3}\mu)(r\sin\theta\,d\varphi)^2, \end{aligned} \tag{80}$$

where

$$\mu = Mr^2/R^3, \tag{81a}$$

and

$$q = 3\cos^2\theta - 1. \tag{81b}$$

Again, R must be considered the function of t given in Eqs (72), or equivalently one may take R as the time coordinate and use Eq. (73) to eliminate dt^2 in favor of dR^2 in Eq. (80).

In the following paper[8] this metric provides boundary conditions for a computation of tidal deformations of a freely falling Schwarzschild singularity (wormhole mouth). It is also evidently well suited to a calculation of tides in an elastic test body whose center of mass would define the geodesic $x^i = 0$. We content ourselves here with a mathematical example and investigate the shape of a sphere. Define a sphere Σ as the surface formed by all points a fixed *proper* distance r measured out orthogonally from some point on the central geodesic. For the coordinates of Eq. (80) this is the surface t = const, r = const, whose metric is, therefore,

$$(ds^2)_\Sigma = (1 + \tfrac{1}{3}\mu)(r\, d\theta)^2 + (1 + \tfrac{1}{3}q\mu - \tfrac{1}{3}\mu)(r \sin\theta\, d\varphi)^2. \tag{82a}$$

From this metric we find that the area of the sphere Σ is just $4\pi r^2$, independent of the small quantity $\mu = Mr^2/R^3$ in first order, but a change in intrinsic shape can be readily computed. The length of a great circle φ = const over the poles of this sphere is

$$L_{\text{poles}} = r \int_0^{2\pi} (1 + \tfrac{1}{3}\mu)^{\frac{1}{2}}\, d\theta \approx 2\pi r(1 + \tfrac{1}{6}\mu). \tag{82b}$$

Similarly, the circumference of the equator, $\theta = \frac{1}{2}\pi$, is

$$L_{\text{equator}} \approx 2\pi r(1 - \tfrac{1}{3}\mu). \tag{82c}$$

As a measure of the distortion of the shape of this sphere, then, we may take

$$\eta = \frac{L_{\text{poles}} - L_{\text{equator}}}{L_{\text{poles}} + L_{\text{equator}}} \approx \frac{\mu}{4} = \frac{Mr^2}{4R^3}. \tag{83}$$

Thus, a sphere r = const is a surface shaped like a football pointing toward the center of gravitation.

IX. RANGE OF VALIDITY OF THE FERMI EXPANSION

In this section we point out that in most situations where the Fermi metric expanded through quadratic terms is a useful description, the time dependence of the metric can be considered adiabatic, that is, time derivatives of the metric will be negligible in comparison to space derivatives. Order of magnitude-wise, the Fermi metric can be written

$$g \simeq 1 + r^2 K(t) + r^3 \left.\frac{\partial K}{\partial r}\right|_G + O(r^4), \tag{84}$$

where r is proper distance normal to the geodesic, t is proper time along the geodesic, and K represents a typical component of the curvature tensor. For this metric, the ratio of time to space derivatives (computed from the r^2 term only) is

$$\frac{\partial g/\partial t}{\partial g/\partial r} = \frac{r}{K}\frac{\partial K}{\partial r}. \tag{85}$$

But if we assume that the quadratic terms in (84) are an adequate approximation to the metric, then the cubic terms must be negligible in comparison to the quadratic ones,

$$\frac{r}{K}\frac{\partial K}{\partial r} \ll 1. \tag{86}$$

This small quantity is almost the one appearing in Eq. (85), except a space and time derivative are interchanged. But as a sort of causality condition, one expects that

$$\partial K/\partial t \lesssim \partial K/\partial r, \tag{87}$$

for in the contrary case, a disturbance would appear spontaneously at some point ($\partial K/\partial t$ large) without having arrived there as a wave propagating with velocity less than $c = 1$. Thus the quadratic Fermi approximation (86) together with causality in the sense of Eq. (87) imply in Eq. (85) the adiabatic condition

$$\partial g/\partial t \ll \partial g/\partial r. \tag{88}$$

These results can be specialized to the Schwarzschild case and give some surprises. For this metric we have from Eq. (78)

$$K = M/R^3. \tag{89}$$

Using Eqs. (74) for $\partial/\partial t$ and $\partial/\partial x$ we can test the causality conditions (87) and find that it reads

$$\frac{R'}{XT'} = \left[\left(\frac{2M}{R} - \frac{2M}{R_0}\right)\Big/\left(1 - \frac{2M}{R_0}\right)\right] \leq 1, \tag{90}$$

which is always *violated* for $R < 2M$. [Other situations which violate the causality condition of Eq. (87) are the expanding-universe cosmological models where one assumes $\partial K/\partial r = 0$.] A condition which will ensure the validity of the Fermi expansion is

$$Kr^2 = Mr^2/R^3 \ll 1, \tag{91}$$

and this can be satisfied by taking (r/R) small enough even if (M/R) is large. Thus the Fermi expansion is useful even inside the Schwarzschild "singularity." The adiabatic condition computed from Eqs. (85) and (74) reads

$$\frac{\partial g/\partial t}{\partial g/\partial r} = \frac{r}{R}R' \simeq \left(\frac{Mr^2}{R^3}\right)^{\frac{1}{2}} \ll 1, \tag{92}$$

and is satisfied as a consequence of Eq. (91) which is a stronger convergence requirement than Eq. (86).