Problem set III for "Boltzmann equation"

Due March 6, 2019

Problem I: Phenomenology of thermal conductivity

Let $\vec{h}$ be the heat transfer vector, i.e., the flow of heat per unit time per unit area normal to the isothermal surface through a given point in the body. Assume the experimental Fourier's heat conduction law

$$\vec{h} = -K\ \vec{\nabla} T\ ,$$

where T is the temperature and K is the coefficient of thermal conductivity. Also, the thermal energy absorbed per unit volume is given by $c\rho_m T$, where c is the specific heat and ρ_m is the mass density.

(a) Make an analogy between the thermal quantities $\vec{h},\ K, T, c, \rho_m$ and the corresponding quantities $\vec{E}$ (electric field), $\vec{j}$ (current density), V (voltage), ρ_q (charge density) of steady currents.

(b) Using the results of (a) find the heat conduction equation.

(c) A pipe of inner radius R_1, outer radius R_2 and constant thermal conductivity K is maintained at an inner temperature T_1 and outer temperature T_2. For a length of pipe L find the rate at which the heat is lost and the temperature inside the pipe in the steady state.

(d) When there is heat flow in a heat conducting material, there is an increase in entropy. Find the local rate of entropy generation per unit volume in a heat conductor of given heat conductivity and given temperature gradient. Compare the result with the entropy production in the process of diffusion.

Problem II: Coefficient of thermal conductivity

Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is maintained at a temperature T_1, while the other plate at $y = w$ is at a different temperature T_2. The gas velocity is zero, so that the initial zeroth order approximation to the one particle density is,

$$f_1^{(0)}(y, \vec{p}) = \frac{n(y)}{\big(2\pi k_B T(y)\big)^{\frac{3}{2}}}\ \exp\left(-\frac{\vec{p}^{\,2}}{2mk_B T(y)}\right)\ .$$

(a) What is the necessary relation between $n(y)$ and $T(y)$ to ensure that the gas is under local mechanical equilibrium?

(b) Using Wicks theorem, or otherwise, show that

$$\langle p^2 \rangle_0 = 3\ (mk_BT)\ , \qquad \langle p^4 \rangle_0 = 15\ (mk_BT)^2\ ,$$

where $p^2 = |\vec{p}|^2$ and $\langle \mathcal{O} \rangle_0$ indicates local averages with the Gaussian weight $f_1^{(0)}$. Use the result $\langle p^6 \rangle_0 = 105\ (mk_BT)^3$ in conjunction with symmetry arguments to conclude

$$\langle p_y^2\, p^4 \rangle_0 = 35\ (mk_BT)^3\ .$$

(c) The zeroth order approximation does not lead to a relaxation of temperature/density variations related as in part (a). Find a better (time independent) approximation $f_1^{(1)}(y, \vec{p})$ using the collision-time approximation for the Boltzmann collision integral (see Problem IV, HW5)

$$\left(\frac{\partial f_1^{(1)}}{\partial t}\right)_{\text{coll}} = -\frac{f_1^{(1)} - f_1^{(0)}}{\tau_K}$$

where τ_K is of the order of the mean time between collisions.

(d) Use $f_1^{(1)}$, along with the averages obtained in part (b), to calculate h_y, the y component of the heat transfer vector $\vec{h}$, and hence find K, the coefficient of thermal conductivity:

$$\vec{h} = -K\ \vec{\nabla} T\ .$$

(e) What is the temperature profile, $T(y)$, of the gas in steady state?