Problem set III for “Boltzmann equation”
Due March 9, 2020

Problem I: Phenomenology of thermal conductivity

Let $\vec{h}$ be the heat transfer vector, i.e., the flow of heat per unit time per unit area normal to the isothermal surface through a given point in the body. Assume the experimental Fourier’s heat conduction law

$$\vec{h} = -K \nabla T,$$

where $T$ is the temperature and $K$ is the coefficient of thermal conductivity. Also, the thermal energy absorbed per unit volume is given by $c_p \rho m T$, where $c$ is the specific heat and $\rho m$ is the mass density.

(a) Make an analogy between the thermal quantities $\vec{h}, K, T, c, \rho m$ and the corresponding quantities $\vec{E}$ (electric field), $\vec{j}$ (current density), $V$ (voltage), $\rho_q$ (charge density) of steady currents.

(b) Using the results of (a) find the heat conduction equation.

(c) A pipe of inner radius $R_1$, outer radius $R_2$ and constant thermal conductivity $K$ is maintained at an inner temperature $T_1$ and outer temperature $T_2$. For a length of pipe $L$ find the rate at which the heat is lost and the temperature inside the pipe in the steady state.

(d) When there is heat flow in a heat conducting material, there is an increase in entropy. Find the local rate of entropy generation per unit volume in a heat conductor of given heat conductivity and given temperature gradient. Compare the result with the entropy production in the process of diffusion.

Problem II: Coefficient of thermal conductivity

Consider a classical gas between two plates separated by a distance $w$. One plate at $y=0$ is maintained at a temperature $T_1$, while the other plate at $y=w$ is at a different temperature $T_2$. The gas velocity is zero, so that the initial zeroth order approximation to the one particle density is,

$$f_1^{(0)}(y,\vec{p}) = \frac{n(y)}{(2\pi k_B T(y))^{\frac{3}{2}}} \exp\left(-\frac{\vec{p}^2}{2mk_B T(y)}\right).$$

(a) What is the necessary relation between $n(y)$ and $T(y)$ to ensure that the gas is under local mechanical equilibrium?
(b) Using Wicks theorem, or otherwise, show that
\[ \langle p^2 \rangle_0 = 3 \, (mk_BT) \quad \text{and} \quad \langle p^4 \rangle_0 = 15 \, (mk_BT)^2 , \]
where \( p^2 = |\vec{p}|^2 \) and \( \langle O \rangle_0 \) indicates local averages with the Gaussian weight \( f_1^{(0)} \). Use the result \( \langle p^6 \rangle_0 = 105 \, (mk_BT)^3 \) in conjunction with symmetry arguments to conclude
\[ \langle p^2 p^4 \rangle_0 = 35 \, (mk_BT)^3 . \]

(c) The zeroth order approximation does not lead to a relaxation of temperature/density variations related as in part (a). Find a better (time independent) approximation \( f_1^{(1)}(y, \vec{p}) \) using the collision-time approximation for the Boltzmann collision integral (see Problem IV, HW5)
\[
\left( \frac{\partial f_1^{(1)}}{\partial t} \right)_{\text{coll}} = -\frac{f_1^{(1)} - f_1^{(0)}}{\tau_K}
\]
where \( \tau_K \) is of the order of the mean time between collisions.

(d) Use \( f_1^{(1)} \), along with the averages obtained in part (b), to calculate \( h_y \), the y component of the heat transfer vector \( \vec{h} \), and hence find \( K \), the coefficient of thermal conductivity:
\[ \vec{h} = -K \, \vec{\nabla} T . \]

(e) What is the temperature profile, \( T(y) \), of the gas in steady state?