Problem set II for "Boltzmann equation"

Due February 27, 2019

Problem I: Collision time

Consider a molecule in a dilute gas. Assuming that the probability of collision events occurring in subsequent time intervals are statistically independent find

(a) $P(t)$ – the probability that the molecule undergoes no collisions for a time t;

(b) $p_{\rm col}(t)\,\mathrm{d}t$ – the probability that the molecule moves without colliding during time t and then collides within the infinitesimal time interval $(t, t + \mathrm{d}t)$.

Express the result in terms of the mean time between the collisions

$$\tau = \langle\, t \,\rangle \equiv \int_0^\infty \mathrm{d}t\, t\, p_{\rm col}(t)$$

(this is also called the "mean free time" or "relaxation time" of the molecule).

Problem II: Mean free path

(a) Assuming that a molecule can be approximated by a hard sphere of diameter d, find the number of collisions $\nu \equiv \tau^{-1}$ that one molecule undergoes per unit time with the other molecules.

(b) Find the mean free path λ.

(c) Estimate the mean free time τ and mean free path λ for air at normal temperature and pressure conditions. Compare λ with the mean intermolecular distance which is is $\ell_{\rm imd} \equiv n^{-\frac{1}{3}}$ where n is the number density.

Problem III: Sutherland's formula

Consider a more realistic situation. The gas particles are approximated as hard spheres of diameter d with an additional weak attractive potential $V(r) = V(|\vec{r}_1 - \vec{r}_2|)$, say, of the form:

$$V(r) = \begin{cases} \infty & \text{for} \quad 0 \le r < d \\ -\frac{\alpha}{r^n} & \text{for} \quad r \ge d \end{cases}$$

where $n > 2,\ \alpha > 0$ (see Fig. 1)

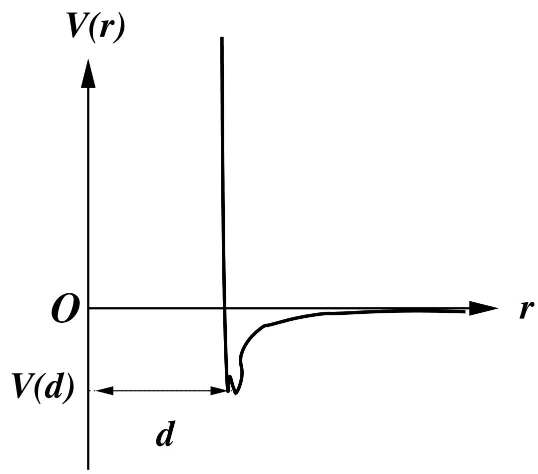

Figure 1:

Show that in this case, the mean free path λ depends on the temperature according to the Sutherland formula

$$\lambda = \frac{\lambda_0}{1 + T_0/T} \, .$$

where T_0 is some positive constant with dimensions of temperature.

Hint: see p. 511 in Sutherland's original paper
https://web.stanford.edu/ cantwell/AA210A_Course_Material/Sutherland_Viscosity_Model.pdf

Problem IV: Collision-time approximation for the Boltzmann equation

Consider an ideal gas of particles with electric charge q and mass m. In the absence of an external electric field, the equilibrium one particle density does not depend on $\vec{r}$ and has the following general form

$$f_1^{(0)} = f(\varepsilon(p)) \ . \tag{1}$$

In the case of a dilute gas

$$f(\varepsilon) = const \ n \ \mathrm{e}^{-\frac{\varepsilon}{k_B T}} \qquad (n = N/V)$$

which coincides with the Maxwell distribution for nonrelativistic dispersion $\varepsilon(p) = \frac{\vec{p}^2}{2m}$. Notice however that eq. (1) can be applied for more general situations, say for the quantum Fermi gas.

In the presence of a weak uniform electric field $\vec{E}$, the distribution function can be derived from Boltzmann's transport equation in the so-called collision-time approximation:

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \right) f_1^{(1)}(\vec{r}, \vec{p}, t) = - \frac{f_1^{(1)} - f_1^{(0)}}{\tau} \equiv \left(\frac{\partial f_1^{(1)}}{\partial t} \right)_{\mathrm{coll}} , \tag{2}$$

where τ is the characteristic time between collisions and $\vec{F} = q\vec{E}$.

(a) As a first-order approximation in terms of the electric field, show that

$$f_1^{(1)}(\vec{p}) = \left(f - \tau q\, \vec{E} \cdot \vec{v}\, \frac{\partial}{\partial \varepsilon}\, f \right)\Big|_{\varepsilon=\varepsilon(\vec{p})} + O(E^2)$$

(b) Use this result and Ohm's law $\vec{j} = \sigma\, \vec{E}$ to show that the conductivity is given by

$$\sigma = \frac{\tau q^2}{3} \int \mathrm{d}^3\vec{p}\; v^2 \left(-\frac{\partial f}{\partial \varepsilon} \right) . \tag{3}$$

(c) Calculate the conductivity in the case of the Maxwell distribution. (Note that for metallic electrons this result is preliminary, since we have to use in (3) the correct Fermi-Dirac distribution to describe equilibrium).