Problem set I for "Boltzmann equation"

Due February 20, 2019

Problem I: Averaging

Consider a gas of N particles whose kinetic and (total) potential energies are given by

$$K = \sum_{a=1}^{N} \varepsilon(\vec{p}_a) \ , \qquad U_{\rm tot} = \sum_{a=1}^{N} U(\vec{r}_a) \ + \sum_{1\le a<b\le N} \mathcal{V}(\vec{r}_a - \vec{r}_b) \ .$$

Note that the dispersion relation $\varepsilon = \varepsilon(\vec{p})$ **does not** necessarily have to be $\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$.

(a) Assuming the one and two particle densities $f_1(\vec{r},\vec{p},t)$, $f_2(\vec{r}_1,\vec{r}_2,\vec{p}_1,\vec{p}_2,t)$ are known, find the ensemble expectation values $\langle K\rangle$ and $\langle U_{\rm tot}\rangle$.

(b) Suppose the gas fills a volume V and there are no other external forces exerted on the gas except from the reservoir walls. Assuming the gas reached an equilibrium state and the corresponding one and two particle equilibrium densities are given,

$$f_1 = f_1(\vec{r},\vec{p}) \ , \quad f_2 = f_2(\vec{r}_1,\vec{r}_2,\vec{p}_1,\vec{p}_2) \ ,$$

show that

$$\boxed{\int \mathrm{d}^3\vec{r}\mathrm{d}^3\vec{p}\, f_1(\vec{r},\vec{p})\ \vec{p}\cdot\frac{\partial\varepsilon(\vec{p})}{\partial\vec{p}} = 3PV + \frac{1}{2}\int \mathrm{d}^3\vec{r}_1\mathrm{d}^3\vec{r}_2\mathrm{d}^3\vec{p}_1\mathrm{d}^3\vec{p}_2\, f_2(\vec{r}_1,\vec{r}_2,\vec{p}_1,\vec{p}_2)\,\vec{r}\cdot\frac{\partial\mathcal{V}(\vec{r})}{\partial\vec{r}}} \qquad (1)$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ and P stands for the gas pressure.
Hint: Use the virial theorem and the ergodic hypothesis.

(c) In the case of the ideal gas the second term on the right hand side of eq.(1) is neglected while the equilibrium densities take the form

$$f_1(\vec{r},\vec{p}) = \frac{N}{V}\ f(p) \ , \qquad f_2(\vec{r}_1,\vec{r}_2,\vec{p}_1,\vec{p}_2) = f_1(\vec{r}_1,\vec{p}_1)f_1(\vec{r}_2,\vec{p}_2) \ , \qquad (2)$$

where $p = |\vec{p}|$. Show that for $\varepsilon = \varepsilon(p)$ and

$$f(p) = const\ \mathrm{e}^{-\frac{\varepsilon(p)}{k_B T}} \ , \qquad (3)$$

the equation of state for the ideal gas is given by

$$\boxed{PV = Nk_BT}$$

independent of the explicit form of the dispersion relation $\varepsilon = \varepsilon(p)$.

(d) Calculate the average value of the interatomic gravitational potential energy,

$$\mathcal{V}_G = -\sum_{1\leq a<b\leq N} \frac{G\,m^2}{|\vec{r}_a - \vec{r}_b|}\ ,$$

(m is the particle mass, G is the gravitational constant) for a gas filling a ball of radius R. Assume the form (2) for the equilibrium one and two particle densities.

Problem II: Relativistic ideal gas

(a) For the ideal gas from Problem IIc, the average kinetic energy $\langle K\rangle_{\rm eq}$ coincides with the internal energy E. Find E in the case of the relativistic dispersion relation

$$\varepsilon(\vec{p}) = \sqrt{p^2c^2 + m^2c^4}\ .$$

Here m is the particle mass and c stands for the speed of light. Express the result in terms of the dimensionless parameter

$$\zeta = \frac{mc^2}{k_B T}$$

and modified Bessel function of the second order:

$$K_s(\zeta) = \int_0^\infty \mathrm{d}\theta\ \cosh(s\theta)\ \mathrm{e}^{-\zeta\cosh(\theta)}\ . \tag{4}$$

Hint: Useful identities

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Big(\zeta^{-1}K_s(\zeta)\Big) = -\frac{s+1}{\zeta^2}\ K_s(\zeta) - \frac{1}{\zeta}\ K_{s-1}(\zeta)\ , \qquad \frac{\mathrm{d}K_s(\zeta)}{\mathrm{d}\zeta} = -\frac{1}{2}\ K_{s+1}(\zeta) - \frac{1}{2}\ K_{s-1}(\zeta)\ . \tag{5}$$

(b) Analyze the nonrelativistic, $\zeta \gg 1$ and ultrarelativistic $\zeta \ll 1$ limits. In particular show that for the massless particles (say photons), the energy density $V^{-1}\,E$ and the pressure P obey the temperature independent relation

$$\boxed{P = \frac{E}{3V} \qquad \text{(for massless particles)}}$$

The latter is widely used in many different problems of physics and astronomy.

(c) Consider the heat capacity defined by the equation

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V\ .$$

Plot the dimensionless heat capacity per particle, $\frac{C_V}{k_B N}$, as a function of the dimensionless parameter $\zeta = \frac{mc^2}{k_B T}$. Identify the limiting value $\frac{C_V}{k_B N}$ at $\zeta \to \infty$. It should be familiar to you from high school physics.

Problem III: Sun temperature

(a) Suppose that the Hamiltonian of a mechanical system is given by

$$H = K(\vec{p}_1, \dots, \vec{p}_N) + U(\vec{r}_1, \dots, \vec{r}_N) \ ,$$

where K is the kinetic energy,

$$K = \sum_{a=1}^{N} \frac{\vec{p}_a^{\,2}}{2m_a} \ ,$$

and the potential energy U is a homogeneous function of the coordinates of degree k. In other words, U satisfies the condition

$$U(\lambda\vec{r}_1, \dots \lambda\vec{r}_N) = \lambda^k \ U(\vec{r}_1, \dots \vec{r}_N) \tag{6}$$

for any constant λ. Using the virial theorem and the ergodic hypothesis show that the ensemble averages of the potential and kinetic energy over the equilibrium state (if this state exists) obey the relations

$$2 \, \langle K \rangle_{\rm eq} = k \, \langle U \rangle_{\rm eq} \ . \tag{7}$$

(b) Suppose that the Sun can be approximated by a system of particles (Hydrogen and Helium atoms) with a total mass $M_S = 2. \times 10^{30}\,$kg interacting according to Newton's gravitational law and moving inside a ball of radius $R_S = 7. \times 10^{8}\,$m. Assume further that the ball has a constant density and temperature. Using this simple model estimate the temperature inside the Sun.

Hint: Use the results of Problems I(d) and II(a,b)

Problem IV: One dimensional gas

A thermalized gas particle is suddenly confined to a one dimensional trap. The corresponding mixed state is described by the initial probability density function

$$\rho(q,p,t)|_{t=0} = \delta(q) \ f(p) \ , \qquad \text{where} \quad f(p) = (2\pi m k_B T)^{-\frac{1}{2}} \ \mathrm{e}^{-\frac{p^2}{2mk_BT}} \ .$$

(a) Starting from Liouville's equation, derive $\rho(q,p,t)$ and sketch it in the (q,p) plane.

(b) Derive the expressions for the averages $\langle q^2 \rangle$ and $\langle p^2 \rangle$ at $t > 0$.

(c) Suppose that hard walls are placed at $q = \pm Q$. Describe $\rho(q,p,t \gg \tau)$, where τ is an appropriate large relaxation time.

(d) A "coarse-grained" density $\tilde{\rho}$ is obtained by ignoring variations of ρ below some small resolution in the (q,p) plane; by averaging ρ over cells of the resolution area. Find $\tilde{\rho}(q,p)$ for the situation in part (c), and show that it is stationary.

Problem V: Explosion of a spherical shell

A monatomic ideal gas of particles of mass m is contained inside a spherical shell of radius R at temperature T and with a constant mass density ρ_0. The shell disappears at instant $t = 0$ and the particles start flying away freely.

(a) Ignoring particle collisions, determine the gas mass density ρ as a function of the time and coordinates. Express the result in terms of the error function:

http://mathworld.wolfram.com/Erf.html

$$\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{d}x\ \mathrm{e}^{-x^2}$$

and the dimensionless variables

$$\xi \equiv \frac{|\vec{r}|}{R} , \qquad \eta \equiv \sqrt{\frac{k_B T}{m}}\ \frac{t}{R} .$$

(b) Plot the dimensionless ratio $\frac{\rho}{\rho_0}$ as a function of $\xi = \frac{|\vec{r}|}{R}$ for $\eta = 0, 0.1, 0.5, 1, 2$.

Problem VI: Two component plasma

Consider a neutral mixture of N ions of charge $+e$ and mass m_+, and N electrons of charge $-e$ and mass m_-, in a volume $V = N/n_0$.

(a) Show that the Vlasov equations for this two component system are,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m_+} \cdot \frac{\partial}{\partial \vec{r}} + e\ \frac{\partial \Phi_{\mathrm{eff}}}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{p}}\right) f_+(\vec{r}.\vec{p}, t) = 0 \\ \left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m_-} \cdot \frac{\partial}{\partial \vec{r}} + e\ \frac{\partial \Phi_{\mathrm{eff}}}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{p}}\right) f_-(\vec{r}.\vec{p}, t) = 0 \end{cases} ,$$

where the effective Coulomb potential is given by

$$\Phi_{\mathrm{eff}}(\vec{r}, t) = \Phi_{\mathrm{ext}}(\vec{r}, t) + e \int \mathrm{d}^3\vec{r}'\ G_C(\vec{r} - \vec{r}')\ \left(f_+(\vec{r}', t) - f_-(\vec{r}', t)\right) .$$

Here Φ_{ext} is a potential set up by external charges, and the Coulomb potential $G_C(\vec{r})$ satisfies the differential equation

$$\triangle G_C(\vec{r}) = 4\pi\ \delta^3(\vec{r}) .$$

(b) Assume that the one particle densities have the stationary forms $f_\pm = n_\pm(\vec{r})\, g_\pm(\vec{p})$. Show that the effective potential satisfies the equation

$$\triangle \Phi_{\mathrm{eff}} = 4\pi\ \rho_{\mathrm{ext}} + 4\pi e \left(n_+(\vec{r}) - n_-(\vec{r})\right) .$$

(c) Further, assuming that the densities relax to the equilibrium Boltzmann weights

$$n_{\pm}(\vec{r}) = n_0 \ \mathrm{e}^{\pm \beta e \Phi_{\mathrm{ext}}(\vec{r})}$$

leads to the self-consistency condition

$$\triangle \Phi_{\mathrm{eff}} = 4\pi \ \Big[\rho_{\mathrm{ext}} + 2 e n_0 \ \sinh\big(\beta e \Phi_{\mathrm{ext}}(\vec{r})\big) \Big]$$

known as the Poisson-Boltzmann equation. Due to its nonlinear form, it is generally not possible to solve the Poisson-Boltzmann equation. By linearizing the exponentials, one obtains the simpler Debye equation

$$\triangle \Phi_{\mathrm{eff}} = 4\pi \ \rho_{\mathrm{ext}} + \lambda_D^{-2} \ \Phi_{\mathrm{ext}} \ .$$

Give an expression for the **Debye screening length** λ_D.

(d) Show that the Debye equation has the general solution

$$\Phi_{\mathrm{eff}}(\vec{r}) = \int \mathrm{d}^3\vec{r}\,' \ G_D(\vec{r} - \vec{r}\,') \ \rho_{\mathrm{ext}}(\vec{r}\,') \, ,$$

where

$$G_D(\vec{r}) = \frac{\mathrm{e}^{-r/\lambda_D}}{r} \qquad (r = |\vec{r}|) \ .$$

(e) A body charged to the potential Φ_0, is submerged into the two component plasma. Determine the Debye screening length assuming that the ions and electron temperatures are different, i.e., $T_+ \neq T_-$.

(f) Give the condition for the self-consistency of the Vlasov approximation, and interpret it in terms of the inter-particle spacing.

(g) Show that the characteristic relaxation time $\left(\tau = \frac{\lambda_D}{\langle v \rangle}\right)$ is temperature independent.