

# Problem set for “Liouville’s theorem”

Due February 13, 2019

## Problem I: Elementary examples

Illustrate Liouville’s theorem using the following examples of mechanical motion:

- (a) elastic collision of two one dimensional particles of masses  $m_1$  and  $m_2$ ,
- (b) for the motion of three non-interacting particles in the presence of a constant gravitational field, whose initial states are described by the three points in phase space

$$A = (p_0, z_0) , \quad B = (p_0, z_0 + a) , \quad C = (p_0 + b, z_0) .$$

## Problem II: Predator-prey model

The simplest model of the biocenosis<sup>1</sup> describing two biological species is the so-called predator-prey model (Volterra model). It yields the following system of differential equations

$$\dot{u}_1 = -a(1 - u_2)u_1 , \quad \dot{u}_2 = b(1 - u_1)u_2 \quad (u_1, u_2 \geq 0)$$

where  $a$  and  $b$  are positive constants.

- (a) Suppose at  $t = 0$  we have a circle of radius  $r$  with the center at point  $(A, B)$  ( $0 < r < A, B$ ) in the  $(u_1, u_2)$  plane. The circle has an area  $\mathcal{A}(0) = \pi r^2$  with respect to the Riemannian measure  $du_1 du_2$ . Consider the time evolution of this domain and its area  $\mathcal{A} = \mathcal{A}(t)$  governed by the Volterra system. Calculate the time derivative  $\dot{\mathcal{A}}$  at  $t = 0$ .
- (b) (**This is an optional problem.**) Write a program for the calculation of  $\mathcal{A}(t)$ . Plot this function for the case  $a = b = 1$ ,  $A = B = 1$  and  $r = 0.5$ .
- (c) Show that by a change of variables,  $u_1 = u_1(q, p)$ ,  $u_2 = u_2(q, p)$ , the Volterra system can be brought to the Hamiltonian form.
- (d) Find equilibrium points and determine if they are stable or unstable. Solve the equations and describe the phase trajectories in the vicinity of the equilibrium points.
- (e) Draw a (qualitative) phase portrait of the system using the variables  $(u_1, u_2)$ .

---

<sup>1</sup>In the paleontological literature, the term distinguishes “life assemblages”, which reflect the original living community, living together at one place and time.

### Problem III: Damped harmonic oscillator

The damped harmonic oscillator is described by the linear system

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -\gamma p - kx,$$

(a) Assuming  $\gamma \ll \omega_0 \equiv \sqrt{\frac{k}{m}}$ , draw the phase trajectories for the damped harmonic oscillator in the  $(x, p)$ -plane.

(b) Find the time dependence of the phase volume in the case under consideration.

(c) Show that the equations of motion for the damped harmonic oscillator can not be written in the Hamiltonian form, i.e., there is no (time independent) change of variables:

$$x = x(Q, P), \quad p = p(Q, P)$$

such that the equations of motion can be written in the form

$$\dot{Q} = \frac{\partial H(Q, P)}{\partial P}, \quad \dot{P} = -\frac{\partial H(Q, P)}{\partial Q}$$

for some Hamiltonian  $H$ .

### Problem IV: Evolution of entropy

The normalized ensemble density is a probability in the phase space  $\Gamma$ . This probability has an associated entropy

$$S(t) = - \int_{\Gamma} d\Gamma \rho(\mathbf{z}, t) \log \rho(\mathbf{z}, t),$$

where  $\mathbf{z} = (z^1, z^2, \dots, z^{2f}) \equiv (q^1, \dots, q^f, p_1, \dots, p_f)$ ,

$$d\Gamma = \prod_{i=1}^f dq^i dp_i = dz^1 \dots dz^n \equiv d^n z \quad .$$

Show that if  $\rho(\mathbf{z}, t)$  satisfies Liouville's equation for a Hamiltonian  $H = H(\mathbf{z})$ , then

$$\frac{dS}{dt} = 0 \quad .$$

## Problem V: Statistical description of a harmonic pendulum

A pendulum executes small harmonic motion with an amplitude  $\alpha$  around an equilibrium point  $\phi = 0$ .

- (a) Find the probability to find the angle  $\phi$  within the infinitesimal interval  $[\phi, \phi + d\phi]$ .  
(b) Find the time averages

$$\overline{\phi^n} = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} dt \phi^n(t), \quad \text{where } n = 1, 2, 3, \dots$$

and also

$$\overline{(e^{-ik\phi})} = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} dt e^{-ik\phi(t)}.$$

- (c) Consider an ensemble of  $N$  non-interacting pendulums whose amplitudes are randomly distributed with all moments of this probability distribution finite. Assuming  $N \rightarrow \infty$ , find the probability distribution function  $p_\infty(\varphi)$  for the random variable

$$\varphi = \frac{1}{\sqrt{N}} \sum_{i=1}^N \phi_i.$$

### Some formulae:

- (\*) Euler integral of the first kind (the beta function)

$$B(a, b) = \int_0^1 dx x^{a-1} (1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b > 0).$$

(see e.g. <http://mathworld.wolfram.com/EulerianIntegraloftheFirstKind.html>)

- (\*\*) Particular values of the Euler  $\Gamma$ -function

$$\Gamma(n+1) = n!, \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2n-1)!!}{2^n}, \quad n = 1, 2, \dots$$

<http://mathworld.wolfram.com/GammaFunction.html>

- (\*\*\*) It is useful to remember that

$$\int_0^\pi \frac{d\theta}{\pi} e^{-iz \cos(\theta)} = J_0(z)$$

where  $J_0(z)$  is the conventional Bessel function of the first kind.

(see <http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html>)

## Problem VI: Lorentz invariance of the Liouville measure<sup>2</sup>

Is the Liouville measure invariant w.r.t. Lorentz transformations?

<sup>2</sup>This problem is optional and will not be graded