Problem set II for "Probability"

Due February 6, 2019

Problem I: Thermoelectric emission (10pt)

Thermoelectric emission is a phenomenon in which electrons are emitted from the surface of a metal. Assuming that (a) the electron emissions are statistically independent events, (b) the probability of the emission of one electron during an infinitesimal time interval $\mathrm{d}t$ is equal to $\lambda \mathrm{d}t$ with some finite constant $\lambda > 0$, determine the probability of the emission of n electrons during time t.

Problem II: Characteristic functions (10pt)

Calculate the characteristic function, the mean, and the variance of the following probability density functions:

(a) Uniform

$$p(x) = \begin{cases} \frac{1}{2a} & \text{for} \quad -a < x < a \\ 0 & \text{otherwise} \end{cases} \;;$$

(b) Laplace

$$p(x) = \tfrac{1}{2a} \, \exp\Big(-\tfrac{|x|}{a} \Big) \;;$$

(c) Cauchy

$$p(x) = \frac{a}{\pi(x^2 + a^2)} \;;$$

(d) Gauss

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \, \mathrm{e}^{-\frac{(x-\lambda)^2}{2\sigma^2}} \;.$$

Problem III: Maxwell-Boltzmann distribution (10pt)

(a) Write the properly normalized Maxwell-Boltzmann distribution $f(v)$ for finding particles of mass m with magnitude of velocity in the interval $[v, v + \mathrm{d}v]$ at temperature $\Theta \equiv k_B T$.

(b) What is the most likely speed at temperature Θ?

(c) What is the average speed?

(d) What is the average square speed?

(e) Determine the probability that the absolute value of the velocity of relative motion $\mathbf{v}' = \mathbf{v}_1 - \mathbf{v}_2$ of two molecules of a monatomic ideal gas belongs to the interval $[v', v' + \mathrm{d}v']$.

Problem IV: One dimensional gas velocity distribution (10pt)

Consider the velocity of a gas particle in one dimension ($-\infty < v < \infty$) characterized by the time-independent probability density function $\rho = \rho(v)$. Similar to Problem V, introduce the entropy functional

$$S[\rho] = -\int_{-\infty}^{\infty} \mathrm{d}v \; \rho(v) \; \log\big(C\rho(v)\big) \; ,$$

where $C > 0$ is an arbitrary constant with the dimensions of speed. Find the density function $\rho_{\mathrm{ext}}(v)$ extremizing $S[\rho]$ and satisfying one of the following constraints

(a) The average speed is fixed, that is $\langle\, |v| \,\rangle = u \;-\;$ fixed;

(b) The average kinetic energy is fixed, that is $\langle\, \frac{mv^2}{2} \,\rangle = \frac{1}{2}\,\Theta \;-\;$ fixed.

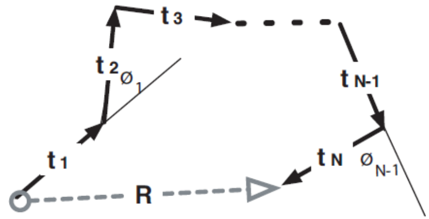
In both cases, find the extremum value $S[\rho_{\mathrm{ext}}]$.

Problem V: Semi-flexible polymer in two dimensions (10pt)

Configurations of a model polymer can be described by either a set of vectors $\{\mathbf{t}_i\}$ of length a in two dimensions (for $i = 1, \ldots, N$), or alternatively by the angles $\{\phi_i\}$ between successive vectors, as indicated in the figure below.
The polymer is at a temperature T, and subject to an energy

$$E = -\kappa \sum_{i=1}^{N-1} \mathbf{t}_i \cdot \mathbf{t}_{i+1} = -\kappa a^2 \sum_{i=1}^{N-1} \cos(\phi_i) \; ,$$

where κ is related to the bending rigidity, and the probability of any configuration is proportional to $\exp(-E/k_B T)$.



(a) Show that $\langle \mathbf{t}_n \cdot \mathbf{t}_m \rangle \propto \exp(-|n-m|\,\xi)$, and obtain an expression for the persistence length $\ell_p = a\,\xi$.

Conventional notation: The Modified Bessel functions of the first kind
(see http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html)

$$I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\phi \; \mathrm{e}^{in\phi + z\cos(\phi)} \quad (n = 0, \pm 1, \pm 2 \ldots)$$

(b) Consider the end-to-end distance $\boldsymbol{R}$ as illustrated in the figure. Obtain an expression for $\langle \boldsymbol{R}^2 \rangle$ in the limit of $N \gg 1$.

(c) Find the probability $p(\mathbf{R})$ in the limit $N \gg 1$.

(d) If the ends of the polymer are pulled apart by a force, the probabilities for polymer configurations are modified by the Boltzmann weight $\exp\left(\frac{\mathbf{F}\cdot\mathbf{R}}{k_B T}\right)$. By expanding this weight, or otherwise, show that

$$\langle \mathbf{R} \rangle = K^{-1}\,\mathbf{F} + O(F^3) \ ,$$

and give an expression for the Hooke's constant K in terms of quantities calculated before.