Problem set II for “Probability”

Due February 10, 2020

Problem I: Thermoelectric emission (10pt)

Thermoelectric emission is a phenomenon in which electrons are emitted from the surface of a metal. Assuming that (a) the electron emissions are statistically independent events, (b) the probability of the emission of one electron during an infinitesimal time interval $dt$ is equal to $\lambda dt$ with some finite constant $\lambda > 0$, determine the probability of the emission of $n$ electrons during time $t$.

Problem II: Characteristic functions (10pt)

Calculate the characteristic function, the mean, and the variance of the following probability density functions:

(a) Uniform

$$p(x) = \begin{cases} \frac{1}{2a} & \text{for } -a < x < a \\ 0 & \text{otherwise} \end{cases} ;$$

(b) Laplace

$$p(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right) ;$$

(c) Cauchy

$$p(x) = \frac{a}{\pi(x^2 + a^2)} ;$$

(d) Gauss

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\lambda)^2}{2\sigma^2}} .$$
Problem III: Maxwell-Boltzmann distribution (10pt)

(a) Write the properly normalized Maxwell-Boltzmann distribution $f(v)$ for finding particles of mass $m$ with magnitude of velocity in the interval $[v, v + dv]$ at temperature $\Theta \equiv k_B T$.

(b) What is the most likely speed at temperature $\Theta$?

(c) What is the average speed?

(d) What is the average square speed?

(e) Determine the probability that the absolute value of the velocity of relative motion $v' = v_1 - v_2$ of two molecules of a monatomic ideal gas belongs to the interval $[v', v' + dv']$.

Problem IV: One dimensional gas velocity distribution (10pt)

Consider the velocity of a gas particle in one dimension ($-\infty < v < \infty$) characterized by the time-independent probability density function $\rho = \rho(v)$. Similar to Problem V, introduce the entropy functional

$$S[\rho] = -\int_{-\infty}^{\infty} dv \rho(v) \log(C\rho(v)),$$

where $C > 0$ is an arbitrary constant with the dimensions of speed. Find the density function $\rho_{ext}(v)$ extremizing $S[\rho]$ and satisfying one of the following constraints

(a) The average speed is fixed, that is $\langle |v| \rangle = u$ – fixed;

(b) The average kinetic energy is fixed, that is $\langle \frac{mv^2}{2} \rangle = \frac{1}{2} \Theta$ – fixed.

In both cases, find the extremum value $S[\rho_{ext}]$.

Problem V: Semi-flexible polymer in two dimensions (10pt)

Configurations of a model polymer can be described by either a set of vectors $\{t_i\}$ of length $a$ in two dimensions (for $i = 1, \ldots, N$), or alternatively by the angles $\{\phi_i\}$ between successive vectors, as indicated in the figure below. The polymer is at a temperature $T$, and subject to an energy

$$E = -\kappa \sum_{i=1}^{N-1} t_i \cdot t_{i+1} = -\kappa a^2 \sum_{i=1}^{N-1} \cos(\phi_i),$$

where $\kappa$ is related to the bending rigidity, and the probability of any configuration is proportional to $\exp(-E/k_BT)$.
(a) Show that $\langle t_n \cdot t_m \rangle \propto \exp(-|n-m| \xi)$, and obtain an expression for the persistence length $\ell_p = a \xi$.

**Conventional notation:** The Modified Bessel functions of the first kind (see [http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html](http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html))

$$I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \ e^{in\phi + z \cos(\phi)} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

(b) Consider the end-to-end distance $R$ as illustrated in the figure. Obtain an expression for $\langle R^2 \rangle$ in the limit of $N \gg 1$.

(c) Find the probability $p(R)$ in the limit $N \gg 1$.

(d) If the ends of the polymer are pulled apart by a force, the probabilities for polymer configurations are modified by the Boltzmann weight $\exp \left( \frac{F \cdot R}{k_B T} \right)$. By expanding this weight, or otherwise, show that

$$\langle R \rangle = K^{-1} F + O(F^3),$$

and give an expression for the Hooke’s constant $K$ in terms of quantities calculated before.