Problem set I for "Probability"

Due January 30, 2019

Problem I: Buffon's[1] needle (10pt)

A needle of length ℓ is dropped at random onto a sheet of paper ruled with parallel lines a distance ℓ apart.

(a) Show that the subjective probability that the needle will cross a line equals $\frac{2}{\pi} = 0.6366\ldots$

(b) Suppose we use the Buffon's needle to estimate a numerical value for π from the objective probability. Estimate the number of drops required to find π with a relative error of 1%.

Check your prediction "experimentally" at https://mste.illinois.edu/activity/buffon/

Problem II: Lucky tickets (10pt)

In many Russian cities you enter the bus, trolley or tram (street car), take a seat and then buy a ticket from the conductor. There is a six digit number printed on each ticket (see Fig.). Some peoples believe in lucky tickets. If the sum of the first three digits on your ticket equals the sum of the last three, your ticket is lucky and you are supposed to eat it.

For a randomly chosen ticket with a number of the form $(n_1 n_2 n_3 n_4 n_5 n_6)$ $(0 \leq n_i \leq 9)$ consider the random variable

$$n_1 + n_2 + n_3 - n_4 - n_5 - n_6$$

which has a set of possible outcomes $S = \{-27, -26, \ldots, 26, 27\}$. Let $p(n)$ be the probability corresponding to the outcome $n \in S$.

(a) Write a code (e.g. in *Mathematica*) for the calculation of $p(n)$. Use it to find $p(0)$ — the probability to get a lucky ticket.

(b) Derive an exact analytic expression for $p(n)$.

(c) Find an exact analytic expression for the probability that the sum of the first two digits, $n_1 + n_2$, is equal to the sum of the remaining four numbers, $n_3 + n_4 + n_5 + n_6$. Check your prediction by means of a numerical calculation similar to (I).

[1] Georges-Louis Leclerc, Comte de Buffon (1707-1788) was a French naturalist, mathematician, cosmologist, and encyclopédiste.

Problem III: Random walk in one dimension (10pt)

Consider a particle in one dimension which performs a step of unit length at each second with the probability a of moving to the right, the probability b of moving to the left, and the probability $1 - a - b$ of staying at the same point. In the initial instant, the particle starts at the origin, $x = 0$. Let $p_N(x)$ be the probability to find the particle at the point x after N seconds. Suppose that the particle can not move to the left of the origin, i.e., $p_N(x) = 0$ for any $x = -1, -2, \ldots$ and any N (if $x = 0$ the particle, rather than making a left step, doesn't move). Assuming $a < b$ find the stationary (equilibrium) probability distribution, i.e.,

$$p_{\rm eq}(x) = \lim_{N\to\infty} p_N(x)$$

and calculate the mean and the variance:

$$\langle\, x \,\rangle_{\rm eq}\,, \qquad \langle\, x^2 \,\rangle_{\rm eq} - \langle\, x \,\rangle_{\rm eq}^2 \,.$$

What happens for the case with $a > b$?

Problem IV: An improvement of the Stirling formula (10pt)

(a) Show that

$$N! = \Gamma(N+1) \quad \text{for} \quad N = 1,\, 2,\, 3\ldots\,,$$

where $\Gamma(x)$ is the Euler Γ- function defined for real $x > 0$ by the convergent integral

$$\Gamma(x) \equiv \int_0^\infty \mathrm{d}t\ t^{x-1}\ \mathrm{e}^{-t}\ .$$

See also http://mathworld.wolfram.com/GammaFunction.html

(b) Using the saddle point approximation establish the Stirling formula:

$$N! \approx \sqrt{2\pi N}\ \left(\frac{N}{\mathrm{e}}\right)^N \qquad \text{for} \quad N \gg 1\ .$$

We will mostly be interested in the case of very large $N \sim 10^{23}$. What is the accuracy of the Stirling approximation for $N = 5$?

(c) Show that

$$N! \asymp \sqrt{2\pi N}\ \left(\frac{N}{\mathrm{e}}\right)^N\ \left(1 + \frac{C}{N} + O(N^{-2})\right) \qquad \text{as} \quad N \to \infty\ ,$$

and find the constant C. How does this correction improve the accuracy for the case $N = 5$.

Problem V: Diffusion (10pt)

Consider an impurity which diffuses within a segment of length L. At $t = 0$, the impurity is located at a distance αL $(0 < \alpha < 1)$ from the left edge of the segment.

(a) Find the probability density function $\rho(x,t)$ for any $t > 0$ ($\rho(x,t)\,\mathrm{d}x$ is equal to the probability of finding the impurity within the infinitesimal segment $[x, x + \mathrm{d}x]$ at instant t). Express the dimensionless function $L\rho$ in terms of the dimensionless variables

$$X = \frac{x}{L}\ , \qquad T = \frac{t}{\tau_{\mathrm{relax}}}\ , \qquad \text{where} \quad \tau_{\mathrm{relax}} = \frac{L^2}{\pi^2 D}$$

and D is the diffusion coefficient.

(b) Plot $L\rho$ as a function of $X \in [0,1]$ for $T = 0.1, 0.5, 1, 2$ and $\alpha = 1/4$.

Conventional notation: The Jacobi Theta function
(see http://mathworld.wolfram.com/JacobiThetaFunctions.html)

$$\vartheta_3(z,q) = \sum_{m=-\infty}^{\infty} q^{m^2}\ \mathrm{e}^{2miz} = \mathrm{EllipticTheta}[3, z, q]$$

(c) Introduce the entropy associated to the probability density $\rho(x,t)$:

$$S(t) = -\int_0^L \mathrm{d}x\ \rho(x,t)\ \log\left(\ell\rho(x,t)\right)\ .$$

Here ℓ is a "microscopic" length (lattice size) whose dimensions make the argument of the logarithm dimensionless. Find the total entropy production during the diffusion process, i.e., $\Delta S = S(+\infty) - S(0)$.