Electromagnetism describes one of the four fundamental forces of nature. It was formulated by Maxwell in terms of the four equations:

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \]
\[ \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \mu_0 \vec{j} \times \vec{B} = \mu_0 \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}. \]

This simple set of equations governs physics over a remarkably wide range of scales. In particular, classical electromagnetism is responsible for almost all physical phenomena that we experience in our daily life such as friction, magnets and radiation. Last semester in Physics 503, you mainly focused on static limits of electromagnetism, which allows us to drop the time-dependent terms in the right hand side of Maxwell’s equations. In this limit, the equations for the electric field \( \vec{E} \) and magnetic field \( \vec{B} \) decouples, and the solutions can be described by either Electrostatics or Magnetostatics. The main goal of these lectures is to learn the diverse physical phenomena that follow from the full-fledged time-dependent Maxwell equations, emphasizing the mathematical structures behind them.

**Spacetime in Classical Physics**

The most fundamental assumptions of physics are probably those which concern the concepts of space and time. In Classical Physics and Special Relativity

\[ \text{Spacetime} = 4D \text{ continuum of “events”} \]

An event is just short-hand for a point in spacetime, and I will use these two terms interchangeably.

In Classical Physics space and time are geometrically separate

\[ \text{Spacetime} = \text{Space } \otimes \text{ Time} \]

Here

\[ \text{Space} = \mathbb{E}^3 \quad (\mathbb{R}^3 \text{ with Euclidean structure}) \]
\[ \text{Time} = \mathbb{R} \]

In what follows we will spend some time discussing the model of ‘space’ in classical physics.
Space

• Our experience of physical space is expressed mathematically in the notion of 3-dimensional Cartesian space $\mathbb{R}^3$:

  (i) The space is composed of points which will be denoted by capital letters:
  
  $\mathbb{R}^3 = \bigcup \{P\}$

  (ii) Each point can be labeled by ordered triples of real numbers – the Cartesian coordinates. There is a one-to-one correspondence between all points in space and all possible triples.

• In this course, the triple of numbers describing a coordinate will be combined in a column:

  $P \leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$.

  The point

  $O \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

  is called the coordinate origin. Any point $P \in \mathbb{R}^3$ can be visualized as an arrow with tip at $P$ and tail at $O$. This is called the radius (position) vector and is denoted by $\vec{r}_P$ or just $\vec{r}$. With some abuse of notation, $\vec{r}_P$ will be identified with the column:

  $\vec{r}_P = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$

• We’ll use a set of basis vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ with the coordinates,

  $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

  Any radius vector $\vec{r}$ is given by a linear combination of $\vec{e}_i$:

  $\vec{r} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \sum_{i=1}^{3} x^i \vec{e}_i$.
• The Einstein summation convention will be used. According to this convention, when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index (called dummy index). With this convention
\[ \vec{r} = x^i \vec{e}_i. \]

Note that \( i \) is not a repeated index in
\[ A_{ik} + B_{ik}, \]
because the two occurrences are in different terms.

• One can define a Euclidean distance (metric) in Cartesian coordinates such that for all \( \forall P, Q \in \mathbb{R}^3 \)
\[ \| P - Q \| = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}, \]
where \( \Delta x^i = x^i_P - x^i_Q. \) The space \( \mathbb{R}^3 \) with this “Euclidean structure” will be denoted by \( \mathbb{E}^3. \)
Curvilinear coordinates

- The Cartesian orthogonal coordinate system is very intuitive and easy to handle. Once an origin has been fixed in space and a set of 3 orthogonal axes are anchored to this origin, any point in space is uniquely determined by 3 Cartesian coordinates $x^i$. Suppose we introduce instead curvilinear coordinates
  \[
  \tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3) \equiv \tilde{x}^i(x).
  \]

At this point we assume only two general properties of these three functions

1. $\tilde{x}^i(x)$ are differentiable (in fact, twice differentiable, as later we will need second derivatives), i.e.
   \[
   \frac{\partial \tilde{x}^i(x)}{\partial x^j}, \frac{\partial^2 \tilde{x}^i(x)}{\partial x^j \partial x^k}
   \]
   exist in some domain $\mathcal{D} \subset \mathbb{R}^3$.
2. The map $x \mapsto \tilde{x}$ from $\mathcal{D}$ to some domain $\tilde{\mathcal{D}}$ is one-to-one. This way, there exists an inverse map $\tilde{x} \mapsto x$. In the corresponding domain $\tilde{\mathcal{D}}$ we have inverse functions
   \[
   x^i = x^i(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \equiv x^i(\tilde{x}).
   \]

- By definition
  \[
  x^i(\tilde{x}(x)) = x^i \quad \text{in} \quad \mathcal{D}, \quad \tilde{x}^i(x(\tilde{x})) = \tilde{x}^i \quad \text{in} \quad \tilde{\mathcal{D}}.
  \]
This implies that the Jacobian of $x \mapsto \tilde{x}$, and the Jacobian of $\tilde{x} \mapsto x$, are both non-vanishing in the appropriate domains

\[
\det \left( \frac{\partial \tilde{x}^i(x)}{\partial x^j} \right) \neq 0 \quad x \in \mathcal{D}, \quad \det \left( \frac{\partial x^i(\tilde{x})}{\partial \tilde{x}^j} \right) \neq 0 \quad \tilde{x} \in \tilde{\mathcal{D}}.
\]

Since

\[
  x^i = x^i(\tilde{x}(x)) \quad \text{and} \quad \tilde{x}^i = \tilde{x}^i(x(\tilde{x})) = \tilde{x}^i
\]

are locally invertible functions, and one is the inverse of another,

\[
x^i(\tilde{x}(x)) = x^i, \quad \tilde{x}^i(x(\tilde{x})) = \tilde{x}^i,
\]

one has

\[
\frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \equiv \delta^i_k.
\]

- Since
  \[
  dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j
  \]
we have for an infinitesimal distance
  \[
ds^2 \equiv dx^i dx^i = \tilde{g}_{ij}(\tilde{x}) d\tilde{x}^i d\tilde{x}^j,
  \]
where

$$\tilde{g}_{jk} = \tilde{g}_{kj} = \frac{\partial x^i(\tilde{x})}{\partial \tilde{x}^j} \frac{\partial x^i(\tilde{x})}{\partial \tilde{x}^k}.$$

The infinitesimal distance $ds^2$ is a positive definite differential quadratic form and $\tilde{g}_{ij}$ can be thought as the entries of a $3 \times 3$ symmetric matrix:

$$\tilde{g} = \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} \end{pmatrix}.$$ 

Each entry, in general, depends on $\tilde{x}$. We’ll refer to $\tilde{g}_{ij}$ as the components of a metric tensor (or simply the metric tensor) in the curvilinear coordinate frame $\tilde{x}$.

- **Example:** Consider the spherical curvilinear coordinates

$$x^1 = r \sin(\theta)\cos(\varphi) \ , \ x^2 = r \sin(\theta)\sin(\varphi) \ , \ x^3 = r \cos(\theta) \ .$$

By direct calculation one finds

$$ds^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2(\theta) \ (d\varphi)^2.$$ 

This way the matrix $\tilde{g}$ reads

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}.$$ 

---

1We’ll usually use the bold face font style for matrices. The italic font is reserved for matrix entries.
Metric-preserving coordinate transformations

• The components of the metric tensor are changed under a coordinate transformation. In Cartesian coordinates they take the simplest possible form:

\[ g = I_{3 \times 3} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \]

We now address the question:

Are there any other sets of curvilinear coordinates where the metric tensor has the same simple form?

• One can consider the problem for more general differential quadratic forms in \( D \)-dimensional Cartesian space \( \mathbb{R}^D \),

\[ \eta_{ij} \, dx^i \, dx^j , \]

such that all the entries of the \( D \times D \) matrix

\[ \eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1D} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2D} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{D1} & \eta_{D2} & \cdots & \eta_{DD} \end{pmatrix} \]

are \( x \)-independent constants. In the case of Euclidean space \( \mathbb{E}^D \), the metric \( \eta \) coincides with the identity matrix \( I_{D \times D} \). Note that, in what follows, we will not necessarily require that \( \eta \) is a positive definite matrix, only that

\[ \det(\eta) \neq 0 . \]

• Performing the transformation \( x \mapsto \tilde{x} \) to arbitrary curvilinear coordinates in \( \mathbb{R}^D \), one has

\[ \eta_{ij} \, dx^i \, dx^j = \tilde{g}_{\nu'\nu} (\tilde{x}) \, d\tilde{x}^{\nu'} \, d\tilde{x}^{\nu'} = \tilde{g}_{\nu'\nu} \Lambda_{i}^{\nu'} \Lambda_{j}^{\nu} \, dx^i \, dx^j . \]

The \( x \)-dependent functions

\[ \Lambda_{i}^{\nu} (x) = \frac{\partial \tilde{x}^{\nu}}{\partial x^i} \]

form the \( D \times D \) matrix \( \Lambda = \Lambda (x) \), which is called the Jacobi (transition) matrix of the map

\[ x \xrightarrow{\Lambda} \tilde{x} . \]

Its determinant is referred to as the Jacobian:

\[ J_{\Lambda} = \det(\Lambda) \neq 0 . \]
We are looking for generalized coordinates such that
\[ \tilde{g}_{ij} = \eta_{ij} : \quad \eta_{ij} \, dx^i dx^j = \eta_{i'j'} \Lambda^i_j' \, dx^i dx^j. \]

In this case \( \tilde{x}^i = \tilde{x}^i(x) \) satisfy a system of non-linear partial differential equations:
\[ \eta_{ij} = \eta_{i'j'} \Lambda^i_j'(x) \Lambda^j_i'(x). \]

Differentiating with respect to \( x^k \) one has:
\[
0 = \eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^k} \Lambda^j_k' + \eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^k} \quad (i)
\]

Interchanging \( i \leftrightarrow k \) and \( j \leftrightarrow k \) in the above equation:
\[
0 = \eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^i} \Lambda^j_k' + \eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^i} \quad (ii)
\]
\[
0 = \eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^j} \Lambda^j_k' + \eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^j} \quad (iii)
\]

Now we subtract (iii) from the sum of (i) and (ii):
\[
\eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^k} \Lambda^j_k' + \eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^k} - \eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^i} = 0.
\]

Since
\[ \Lambda^i_j' = \frac{\partial \tilde{x}^i}{\partial x^j} : \quad \frac{\partial \Lambda^i_j'}{\partial x^k} = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} = \frac{\partial \Lambda^i_j'}{\partial x^i}, \]

one has
\[
\eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^k} - \eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^j} = 0
\]
\[
\eta_{i'j'} \Lambda^i_j' \frac{\partial \Lambda^j_k'}{\partial x^i} - \eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^j} \Lambda^j_k' = 0
\]
\[
\eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^k} \Lambda^j_k' = \eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^j} \Lambda^j_k'
\]

and, therefore,
\[
2 \eta_{i'j'} \frac{\partial \Lambda^i_j'}{\partial x^k} \Lambda^j_k' = 0.
\]

Taking into account that \( \det(\eta_{ij}) \neq 0 \) and \( \det(\Lambda^i_j') \neq 0 \) one can transect first by \( \Lambda \) and then \( \eta^{-1} \) yielding
\[
\frac{\partial \Lambda^i_j}{\partial x^k} = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} = 0.
\]

This implies that
\[ \tilde{x}^i = x'^i \quad \text{with} \quad x'^i \equiv \Lambda^i_j' x^j - a^i, \]
where \( L_j^i \) and \( a^i \) are some \( x \)-independent constants. Note that \( a^j \) are completely arbitrary real numbers, while the \( x \)-independent Jacobi matrix \( \Lambda_j^i \) must satisfy the condition

\[
\Lambda_i^{i'} \eta_{i'j'} \Lambda_j^{j'} = \eta_{ij} .
\]

- For given square matrices \( A \) and \( B \) the matrix product

\[
C = A B
\]

is a square matrixes with elements\(^2\)

\[
C_i^j = A_k^i B_k^j
\]

With the matrix notation the condition \( \Lambda_i^{i'} \eta_{i'j'} \Lambda_j^{j'} = \eta_{ij} \) takes the form

\[
\Lambda^T \eta \Lambda = \eta .
\]

Here the transposed matrix \( \Lambda^T \) is obtained from \( \Lambda \) by interchanging rows and columns:

\[
(\Lambda^T)_j^i \equiv \Lambda_i^j .
\]

\(^2\)In the notation \( C_i^j \) the lower index \( i \) enumerates the columns of \( C \), while the upper index \( j \) enumerates the rows:

\[
C_i^j = \begin{pmatrix}
C_1^1 & C_1^2 & C_1^3 \\
C_2^1 & C_2^2 & C_2^3 \\
C_3^1 & C_3^2 & C_3^3
\end{pmatrix} ,
\]

lower index \( i \) ↔ columns

upper index \( j \) ↔ rows.
Translation and orthogonal transformations

Let us apply the general result obtained above to three dimensional Euclidean space,

\[ \mathbb{E}^3 : \quad D = 3 , \quad \eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \]

Translation of the origin

The transformation

\[ (x')^i = x^i - a^i \]

corresponds to a translation of the origin of the coordinate system. Indeed let us consider two Cartesian coordinate systems related by the translation along some vector \( \overrightarrow{OO'} \):

Let

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  x'_1 \\
  x'_2 \\
  x'_3
\end{pmatrix}
\]

be coordinates of some point \( P \in \mathbb{E}^3 \) w.r.t. coordinate systems \( OXYZ \) and \( O'X'Y'Z' \), respectively. Then

\[
\vec{r} = \vec{r'} + \overrightarrow{OO'} \quad \text{or} \quad x^i = x'^i + a^i \quad (i = 1, 2, 3) ,
\]

where \( a^i \) are coordinates of the constant vector \( \overrightarrow{OO'} \):

\[
\overrightarrow{OO'} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} .
\]
Orthogonal transformations

In the case of \( \mathbb{E}^3 \), the matrix \( \eta \) coincides with \( I_{3\times3} \). Thus the Jacobi matrix for the transformation

\[
x \xrightarrow{\Lambda} x'
\]

preserving the form of the Euclidean metric must be a constant orthogonal matrix

\[
\Lambda^T\Lambda = I_{3\times3}.
\]

Let’s recall the geometrical interpretation of the coordinate transformation

\[
x'^{i} = \Lambda^{i}_{j} x^{j}.
\]

• Even if the origin of the Cartesian frame is fixed we still have an ambiguity in the orientation of the coordinate axes. The orientation is uniquely defined by an ordered triple of orthonormal vectors \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) along the coordinate axes OXYZ:

One can specify a new (primed) Cartesian coordinate system OX’Y’Z’ with the same origin O by choosing another triple \( \{\vec{e}'_1, \vec{e}'_2, \vec{e}'_3\} \):

• The orientation of the primed coordinate system w.r.t. the unprimed one is defined by a set of nine real numbers (direction cosines),

\[
S_{j}^{i} = \vec{e}'_{i} \cdot \vec{e}_{j} = \cos(\vec{e}'_{i} \cdot \vec{e}_{j}) ,
\]

that allow one to express \( \{\vec{e}'_{i}\} \) in terms of \( \{\vec{e}_{i}\} \):

\[
\vec{e}'_{i} = \sum_{j=1}^{3} \vec{e}_{j}S_{j}^{i} \equiv \vec{e}_{j}S_{j}^{i} \quad (i = 1, 2, 3).
\]
Using matrix notations the last equation can be rewritten as

\[
(\vec{e}'_1, \vec{e}'_2, \vec{e}'_3) = (\vec{e}_1, \vec{e}_2, \vec{e}_3) S,
\]

where

\[
S = \begin{pmatrix}
S^1_1 & S^1_2 & S^1_3 \\
S^2_1 & S^2_2 & S^2_3 \\
S^3_1 & S^3_2 & S^3_3 \\
\end{pmatrix}.
\]

- As usual in Linear Algebra, one can associate a linear operator (map) to the matrix \( S \). It transforms the orthonormal basis \( \{ \vec{e}_i \} \) to the primed basis \( \{ \vec{e}'_i \} \). Symbolically,

\[
\vec{e}'_i = \hat{S} \vec{e}_i.
\]

Such a map is called an orthogonal transformation.

- The same radius vector \( \vec{r} \) can be expanded in the two different bases:

\[
\vec{r} = x^j \vec{e}_j = x'^i \vec{e}'_i = x'^i S^j_i \vec{e}_j.
\]

Therefore

\[
x^j = S^j_i x'^i, \quad \text{or equivalently} \quad \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = S \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}
\]

This implies that

\[
S^j_i = \frac{\partial q^j}{\partial x^i} \quad \text{with} \quad q^j = x'^j,
\]

i.e., \( S \) is the transition matrix for the map \( x' \mapsto x \). As for the inverse map \( x \mapsto x' \), one has

\[
x'^i = \Lambda^i_j x^j, \quad \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \Lambda \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},
\]

with

\[
\Lambda = S^{-1}
\]

To summarize, we will employ the notation

\[
x \xrightarrow{\Lambda} x', \quad x' \xrightarrow{S} x.
\]

- The direction cosines are not independent real numbers. Invariance of length,

\[
x^i x'^i = x^j x'^j = S^j_i x^i S^j_p x'^p,
\]

implies

\[
S^j_i S^j_p = \delta_{ip} \quad \text{or} \quad SS^T = I.
\]
Euler theorem

- Recall an important property of orthogonal matrices. Their determinant must be equal to either +1 or -1:

\[(\det S)^2 = 1.\]

This follows from the orthogonality condition and elementary properties of the matrix determinant:

\[\det(AB) = \det(A) \det(B), \quad \det(A^T) = \det(A).\]

Orthogonal transformations with determinant

\[\det S = +1\]

are said to be proper, while those with determinant -1 are called improper.

- Let’s consider some examples of orthogonal transformations.

(i) Rotation about \(OZ\) axis by an angle \(\phi\):

In this case,

\[
\begin{aligned}
\vec{e}'_1 &= \cos(\phi) \vec{e}_1 + \sin(\phi) \vec{e}_2 \\
\vec{e}'_2 &= -\sin(\phi) \vec{e}_1 + \cos(\phi) \vec{e}_2 \\
\vec{e}'_3 &= \vec{e}_3.
\end{aligned}
\]

Then the corresponding transition matrix

\[
S = \begin{pmatrix}
\cos(\phi) & -\sin(\phi) & 0 \\
\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{pmatrix} : \quad \det S = +1 \quad (\star)
\]

Since the determinant is equal to +1, the rotation about the \(OZ\) axis is a proper orthogonal transformation. Also notice that the rotation transforms the right-handed triple \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) to the right-handed triple \((\vec{e}'_1, \vec{e}'_2, \vec{e}'_3)\):\(^3\)

Rotation : right – handed \((\vec{e}_1, \vec{e}_2, \vec{e}_3) \mapsto \) right – handed \((\vec{e}'_1, \vec{e}'_2, \vec{e}'_3)\).

\(^3\)If you form the first three fingers of your right hand into three perpendicular vectors, and point your thumb in the direction of the \(OX\) axis, and your index finger in the direction of the \(OY\) axis, your middle finger will point in the direction of the \(OZ\) axis.
(ii) Rotary reflection (improper rotation) about \( OZ \) axis.

The rotary rotation is a composition of the rotation about the \( OZ \) axis followed by a reflection in the \( OXY \) plane.

\[
\begin{align*}
\vec{e}_1' &= \cos(\phi) \vec{e}_1 + \sin(\phi) \vec{e}_2 \\
\vec{e}_2' &= -\sin(\phi) \vec{e}_1 + \cos(\phi) \vec{e}_2 \\
\vec{e}_3' &= -\vec{e}_3.
\end{align*}
\]

The corresponding transition matrix is given by

\[
S = \begin{pmatrix}
\cos(\phi) & -\sin(\phi) & 0 \\
\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & -1
\end{pmatrix} : \quad \det S = -1 \quad (**)\]

This is an example of an improper orthogonal transformation. Notice that the rotary reflection transforms the right-handed triple \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) to the left-handed triple \((\vec{e}_1', \vec{e}_2', \vec{e}_3')\).

Rotary reflection : right-handed \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) \(\rightarrow\) left-handed \((\vec{e}_1', \vec{e}_2', \vec{e}_3')\).

\[\textbf{Theorem (Euler)}: \text{ An arbitrary 3D orthogonal transformation is either a rotation or a rotary reflection.}\]

This famous statement from Linear Algebra means that for any \(3 \times 3\) orthogonal matrix \( S \) one can always find an orthonormal basis,

\[
\vec{E}_i = \vec{e}_j C^j_i \quad (i = 1, 2, 3)
\]

such that the corresponding orthogonal transformation is a rotation (if \( \det S = +1 \)) or rotary reflection (if \( \det S = -1 \)) about the \( \vec{E}_3 \) direction:

\[
C^{-1} = C^T : \quad C^{-1} S C = \text{canonical form (⋆) or (**) }.
\]

The proof can be found in Goldstein’s textbook Chapter 4, §4.6.
Isometries (symmetries) of $\mathbb{E}^3$

Our previous discussion deals with coordinate transformations, i.e., the freedom in relabeling points in space. This does not correspond to transformations of the space itself. However, the statement concerning the metric preserving coordinate transformations turns out to be of great importance for the geometry of $\mathbb{E}^3$.

- As an illustration, consider translations of the origin:

$$x^i = x'^i + a^i .$$

Define a geometrical transformation of $\mathbb{E}^3$ such that an arbitrary point

$$P = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

maps to the point $P'$ whose coordinates w.r.t. the frame $OXYZ$ coincides with

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

Let us emphasize that this interpretation of $x'^i$ is different than the original one. Previously the triple $\{x'^i\}$ was regarded as coordinates of the same point $P$ w.r.t. prime coordinate system $O'X'Y'Z'$ (so called passive point of view). Now $\{x'^i\}$ is understood as coordinates of a different point $P'$ w.r.t. the original frame $OXYZ$ (active interpretation):

$$P' = \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

This defines a map

$$P \in \mathbb{E}^3 \mapsto P' \in \mathbb{E}^3$$

which preserves Euclidean distances between points

$$P \mapsto P' , \quad Q \mapsto Q' : \quad \|P - P'\| = \|Q - Q'\| .$$

The same construction can repeated in the case of orthogonal transformations.

- In geometry, an Euclidean space isometry is a way of transforming $\mathbb{E}^3$ that preserves geometrical properties such as length. We may conclude that isometries of $\mathbb{E}^3$ are translations, rotation and reflections together with all possible compositions of these basic transformations. A similar statement holds true for multidimensional Euclidean space $\mathbb{E}^D$. 

14
• In physics the invariance w.r.t. translations is usually referred to as **homogeneity** of space (physics doesn’t change (it’s symmetric) under space translations). The invariance w.r.t. to orthogonal transformations is called **isotropy** (different directions around a point are all equivalent).

Let us emphasize again that an arbitrary curvilinear coordinate transformation does not correspond to any geometrical transformation of $\mathbb{E}^3$. It’s just a different way of labeling points in some domain $\mathcal{D}$ of Euclidean space. Any physically meaningful quantity can not depend on the choice of such labeling.