Problem set “Euclidean tensors”

Due February 6, 2023

Problem I

(i) Calculate the integrals
\[ \int_{S^2} d\Omega \, n^i n^j, \quad \int_{S^2} d\Omega \, n^i n^j n^k, \quad \int_{S^2} d\Omega \, n^i n^j n^k n^m \]

over the sphere $S^2$ of unit radius. Here $(n^1, n^2, n^3)$ are the components of a unit vector $\vec{n} \in S^2$.

(ii) Show that $\epsilon_{ijk} \epsilon_{kmn}$ is a rank 4 invariant tensor and find the numerical coefficients in the formula
\[ \epsilon_{ijk} \epsilon_{kmn} = A \delta_{ij} \delta_{mn} + B \delta_{im} \delta_{jn} + C \delta_{in} \delta_{jm}. \]

(iii) Show that $\epsilon_{ijk} \epsilon_{lmn}$ is a rank 6 invariant tensor and prove the relation
\[ \epsilon_{ijk} \epsilon_{lmn} = \text{det} \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix} = \delta_{il} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) - \delta_{im} (\delta_{jl} \delta_{kn} - \delta_{jn} \delta_{kl}) + \delta_{in} (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}). \]
Problem II

(a) Let $T_{j_1 \ldots j_\ell}$ be a totally symmetric and traceless tensor of rank $\ell$,
\[
T_{j_1 \ldots j_\ell} : T_{j_p \ldots j_q \ldots} = T_{j_{p'} \ldots j_{q'} \ldots} \quad \& \quad \delta^{j_p j_q} T_{j_{p'} \ldots j_{q'} \ldots} = 0 \quad \forall p, q \quad (\star)
\]
and $\vec{n}$ be a unit vector, $|\vec{n}| = 1$. Assuming that $T_{j_1 \ldots j_\ell}$ is a constant tensor (i.e., it is the same at any point in space) consider the non-singular function on the sphere
\[
\Psi^{(T)}_{\ell}(\vec{n}) = T_{j_1 \ldots j_\ell} n^{j_1} \ldots n^{j_\ell}.
\]
Show that $\Psi^{(T)}_{\ell}(\vec{n})$ satisfies the equation
\[
-\nabla^2_{\vec{n}} \Psi^{(T)}_{\ell}(\vec{n}) = \ell (\ell + 1) \Psi^{(T)}_{\ell}(\vec{n}),
\]
where $\nabla^2_{\vec{n}}$ is the Laplacian on the round sphere (i.e. $\nabla^2_{\vec{n}}$ is the spherical part of the Laplacian in $\mathbb{E}^3$).

(b) Show that the number of linearly independent components of a totally symmetric and traceless tensor of rank $\ell$ is equal to $2\ell + 1$. Thus, for given $\ell = 0, 1, 2, \ldots$ the functions $\Psi^{(T)}_{\ell}(\vec{n})$ form a linear space of dimensions $2\ell + 1$.

(c) You should know that for given $\ell$, the spherical harmonics
\[
Y_{\ell,m}(\vec{n}) \quad \text{with} \quad m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell
\]
form a linear basis in the space of regular solutions of $\nabla^2_{\vec{n}} \Psi = \ell (\ell + 1) \Psi$. It follows from (a) and (b) that any spherical harmonic can be written in the form
\[
Y_{\ell,m}(\vec{n}) = T^{(m)}_{j_1 \ldots j_\ell} n^{j_1} \ldots n^{j_\ell},
\]
where $T^{(m)}_{j_1 \ldots j_\ell}$ are a certain set of (complex) numbers satisfying the conditions $\star$. Find explicit expressions for $T^{(m)}_{j_1 \ldots j_\ell}$ when $m = -1, 0, 1$ and $T^{(m)}_{j_k}$ when $m = -2, -1, 0, 1, 2$.

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1 Recall that if $\vec{n}$ is parameterized by polar and azimuthal angles $\theta$ and $\varphi$ as
\[
\vec{n} = \begin{pmatrix} n^1 \\ n^2 \\ n^3 \end{pmatrix} = \begin{pmatrix} \sin(\theta) \cos(\varphi) \\ \sin(\theta) \sin(\varphi) \\ \cos(\theta) \end{pmatrix} \in \mathbb{S}^2,
\]
then
\[
\nabla^2_{\vec{n}} = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2}.
\]
Problem III

Consider the $2 \times 2$ unitary matrices with unit determinant:

$$UU^\dagger = I_{2 \times 2} \quad \text{and} \quad \det U = 1.$$ 

(a) Show that the set of all such matrices $U$ form a group. The latter is denoted by $SU(2)$.

(b) Show that any $SU(2)$ matrix can be written in the form

$$U = \begin{pmatrix} \rho_0 + i\rho_3 & \rho_2 + i\rho_1 \\ -\rho_2 + i\rho_1 & \rho_0 - i\rho_3 \end{pmatrix},$$

where the 4 real numbers $(\rho_0, \rho_1, \rho_2, \rho_3)$ satisfy the condition

$$\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 1.$$ 

In other words there is a one-to-one correspondence between elements of the group $SU(2)$ and points on the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$ of unit radius.\(^2\)

(c) Show that the matrix exponential

$$\exp \left( \frac{1}{2} \phi^k \sigma_k \right) \quad (\phi^k = \phi n^k),$$

where $\sigma_k$ are conventional Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an $SU(2)$ matrix and express the corresponding set $(\rho_0, \rho_1, \rho_2, \rho_3)$ through the rotational angle $\phi \sim \phi + 2\pi$ and three dimensional unit vector $\vec{n} \in \mathbb{R}^3$: $|\vec{n}| = 1$. Compare the result with the Euler parameters from Problem IV (HW1).

(d) By the similarity transformation any $4 \times 4$ matrix $U \otimes U$ can be brought to the block diagonal form:

$$U \otimes U = C \begin{pmatrix} 1 & O \\ O & [S^{-1}] \end{pmatrix} C^{-1}.$$ 

Here $S$ is the $3 \times 3$ matrix of finite rotations from Problem IV (HW1) and the $4 \times 4$ matrix $C$ is the same for any $SU(2)$ matrix $U$. Explain why.

(e) Find the explicit form of the $4 \times 4$ constant matrix $C$.

\(^2\)The sphere $S^3$ is one the simplest examples of a mathematical object called a manifold. Hence the group $SU(2)$ possesses the structure of a manifold. Such groups are called continuous groups or Lie groups. The group $SO(3)$ is another example of a Lie group. It has the structure of the 3-dimensional real projective space $\mathbb{RP}^3$. The later can be understood as a 3-dimensional sphere $S^3 \subset \mathbb{R}^4$ defined by the equation $\rho_0^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 1$ with each pair of points $(\rho_0, \rho_1, \rho_2, \rho_3)$ and $(-\rho_0, -\rho_1, -\rho_2, -\rho_3)$ identified (glued together).