Ph 442 Solutions for Problem Set 6

1. The delay of pulsar pulses as a function of frequency provides an way to crudely estimate the distance to the pulsar, which is otherwise difficult to determine. Alternatively, if the distance to the pulsar is known — through its association with a supernova remnant, for example — the pulse dispersion provides an estimate of the average electron density in the interstellar medium.

The dispersion relation relates the wavenumber, k, and frequency, ω, for propagating electromagnetic waves. Waves traveling through a plasma have the relation

$$\omega^2 = \frac{c^2k^2}{1-\omega_p^2/\omega^2} \Rightarrow k^2 = \left(\frac{1}{c^2}\right)\omega^2(1-\omega_p^2/\omega^2) = \frac{1}{c^2}\left(\omega^2-\omega_p^2\right). \tag{1}$$

Here ω_p is the plasma frequency given by $\omega_p^2 = 4\pi n_e e^2/m_e$, where n_e is the electron density, e is the electronic charge, and m_e is the electron mass. The group velocity of the waves is $v_{group} = \partial\omega/\partial k$. It is simpler to calculate $\partial k/\partial\omega$:

$$\begin{aligned} \frac{\partial k}{\partial \omega} &= \frac{\partial}{\partial\omega}\left(\frac{1}{c}\sqrt{\omega^2-\omega_p^2}\right) & (2)\\ &= \left(\frac{1}{2c}\right)\frac{2\omega}{\sqrt{\omega^2-\omega_p^2}} & (3)\\ \Rightarrow v_{group} &= \frac{1}{\partial k/\partial\omega} = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}. & (4) \end{aligned}$$

At frequencies higher than the plasma frequency, the group velocity is smaller than c. For $n_e = 1\ \mathrm{cm}^{-3}$, $\nu_p = \omega_p/(2\pi) = 8.96\times10^3$ Hz. Astronomical observations take place at frequencies much larger than this.

The relation between the distance traveled by a pulse and the time it takes is given by integrating $dt = ds/v_{group}$:

$$\begin{aligned} \int_0^{t_{travel}} dt &= \int_0^d \frac{ds}{v_{group}} & (5)\\ &= \int_0^d \frac{ds}{c\sqrt{1-(\omega_p/\omega)^2}} & (6)\\ \Rightarrow t_{travel} &= \frac{1}{c}\int_0^d ds\left(1+\frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2+\ldots\right) & (7)\\ &\approx \frac{d}{c}+\frac{2\pi e^2}{cm_e\omega^2}\int_0^d n_e ds & (8) \end{aligned}$$

The integral of n_e along the line of sight is traditionally called the dispersion measure.

2. Assume that the Sun collapses conserving angular momentum. The angular momentum of a star is $L = I\omega$, where ω is the angular frequency of rotation and I is the moment of inertia. $I = aMR^2$, where M and R are the mass and radius of the star and a is a constant that depends on the structure of the star ($a = 2/5$ for spherical star with constant density). If the star collapses homologously, *i.e.* the shape of the density profile stays the same, then the a in the moment of inertia is constant. Even if the collapse is not homologous, a will not change that much. Thus, the initial and final angular frequencies are related to the initial and final radii by $aMR_f^2\omega_f = aMR_i^2\omega_i \Rightarrow \omega_f/\omega_i = (R_i/R_f)^2$. Since the relation between the period and angular frequency is $\omega = 2\pi/P$, $P_f/P_i = (R_f/R_i)^2$.

The radius of the Sun is 7.0×10^{10} cm and its equatorial rotation period is 24.5 days (the Sun is a gaseous body and rotates slower at its poles than at the equator). Thus, if the Sun were to collapse to a radius 10 km = 10^6 cm while conserving angular momentum it would have rotation period of

$$P_f = (24.5 \text{ days}) \left(\frac{1.0 \times 10^6 \text{ cm}}{7.0 \times 10^{10} \text{ cm}}\right)^2 = 5.0 \times 10^{-9} \text{ day} = 4.3 \times 10^{-4} \text{ s.} \tag{9}$$

The fastest spinning pulsars have periods of a few milli-seconds, which is a little longer than the above value. The cores of the massive stars which collapse to form neutron stars are apparently either spinning somewhat slower or have a smaller radius than the Sun.

3. The magnetic flux is the stength of the magnetic field times the area threaded by that flux. Thus, the conservation of flux for the Sun says that BR^2 is a constant. The value of the average magnetic field for the Sun is a complex question because the field at the surface is very inhomogenous. This is a consequence of the solar magnetic field being generated much closer to the surface than is the case for the planets. The dipole part of the field also changes in strength roughly sinusoidally, actually reversing polarity every 11 years. A reasonable measure of the dipole component of the solar field is the strength of the field at the poles of the Sun. I found a plot of measurements made Wilcox Solar Observatory at http://wso.stanford.edu/gifs/Polar.gif. The peak field strength is about 1 Gauss and this agrees with values I found in other references. This is about twice the strength of the Earth's magnetic field. In sunspots the field can be about 1000× stronger than the average field. Anyway, if the Sun were to shrink to a radius of 10 km while conserving magnetic flux, the field strength would be

$$B_f = (1 \text{ G}) \left(\frac{7.0 \times 10^{10} \text{ cm}}{1.0 \times 10^6 \text{ cm}}\right)^2 = 4.9 \times 10^9 \text{ G.} \tag{10}$$

This is about 100× smaller than the magnetic field in the typical young pulsar, but different stars have different field strengths and so the calculation does suggest that collapse is a plausible mechanism for producing magnetic fields in neutron stars. However, some researchers suggest that all neutron star magnetic fields could be produced by dynamos after the collapse.

4. a) If the emission from the gamma-ray bursts is beamed, then we see only a fraction of the bursts actually occurring. That fraction will be the fraction of the sky covered by the two

beams from each burst, which is

$$f = \left(\frac{1}{4\pi}\right)\left(2\int_0^{\theta_j} 2\pi\sin(\theta)d\theta\right) = 1-\cos(\theta_j) \simeq \frac{1}{2}\theta_j^2. \tag{11}$$

The last step uses the Taylor expansion for cosine and assumes that $\theta_j \ll 1$. If $\theta_j = 4^\circ = 0.070$ rad, then $f = 2.44\times10^{-3}$ and the observed 2 bursts per day implies a minimum of 820 bursts per day.

b) The typical long gamma-ray burst has a total emitted energy of $E = 10^{51}$ ergs. If this energy is directed into the solid angle $d\Omega$, then the fluence at a distance d is $\mathcal{F} = E/(d\Omega d^2)$. Now $d\Omega = 4\pi f$, where f is the fraction calculated in part a). Thus, the fluence observed at a distance of 10 kpc is

$$\mathcal{F} = \frac{10^{51}\text{ erg}}{4\pi(2.44\times10^{-3})(10\text{ kpc}\times3.086\times10^{21}\text{ cm kpc}^{-1})^2} = 3.4\times10^7\text{ erg cm}^{-2}. \tag{12}$$

This fluence is much larger than the typical 10^{-6} erg cm^{-2} of a gamma ray burst, but these are at much larger distances: >5 Gpc. The largest fluence of gamma rays observed so far from a burst is 1.4 erg cm^{-2}, received from a neutron star in our Galaxy with extremely strong magnetic fields: SGR–1806 on 27 December 2007.

The flux of energy that the Earth receives from the Sun is

$$F_\odot = \frac{L_\odot}{4\pi d^2} = \frac{3.85\times10^{33}\text{ erg s}^{-1}}{4\pi(1.496\times10^{13}\text{ cm})^2} = 1.37\times10^6\text{ erg cm}^{-2}\text{ s}^{-1}. \tag{13}$$

Thus, the calculated fluence from a Galactic gamma ray burst is 25× more than the Earth receives from the Sun at all wavelengths in one second and might have severe effects on terrestrial life. The gamma-rays would be absorbed by the Earth's atmosphere, but associated ultraviolet light would penetrate more deeply and it is likely that the ozone layer would be eliminated for a time. If gamma-ray bursts occur in our Galaxy at the rate given on page 276 of the text, then the Earth should have been illuminated by a burst about once in its 5 billion year history. However, the rate of gamma ray bursts in galaxies with luminosities as large as our own is still poorly known.