Ph 442 Solutions for Problem Set 5

1. In an isothermal shock, the energy jump condition is replaced by the relation that the temperature ahead of and behind the shock is the same, $T_0 = T_1$. The perfect gas law, $P = \rho kT/\mu$ where μ is the mean molecular weight, then says that the pressure is linearly related to the density and so $P_0/\rho_0 = P_1/\rho_1$. This last relation assumes that $\mu_0 = \mu_1$, which is reasonable if the gas has the same temperature and, hence, the same stage of ionization. If the gas remains isothermal even for the pressure and temperature variations associated with sound waves, then $c = (P/\rho)^{1/2}$. It is probably more realistic to assume adiabatic conditions for sound waves, but this would result in only a slightly different value for the sound speed: $(5P/3\rho)^{1/2}$ for a monoatomic ideal gas. The following derivation assumes the isothermal sound speed, but assuming the adiabatic value would yield the same equations with $\mathcal{M}_0$, the preshock Mach number, replaced by $\sqrt{5/3}\mathcal{M}_0$. However, the numerical values of v_1 and ρ_1 are actually unchanged because c is larger by $\sqrt{5/3}$.

a) We still have the momentum jump condition

$$P_0 + \rho_0 v_0^2 = P_1 + \rho_1 v_1^2 \;\Rightarrow\; P_0\left(1 + \frac{v_0^2}{P_0/\rho_0}\right) = P_1\left(1 + \frac{v_1^2}{P_1/\rho_1}\right) \tag{1}$$

$$\Rightarrow\; \frac{P_0}{P_1}\left(1 + \mathcal{M}_0^2\right) = \left(1 + \frac{v_1^2}{c^2}\right) \tag{2}$$

$$\Rightarrow\; \frac{\rho_0}{\rho_1}\left(1 + \mathcal{M}_0^2\right) = \left(1 + \frac{v_1^2}{c^2}\right). \tag{3}$$

The last step above used the isothermal equation of state. The mass jump condition requires that $\rho_0/\rho_1 = v_1/v_0$. Thus, equation 3 becomes

$$\frac{v_1}{v_0}\left(1 + \mathcal{M}_0^2\right) = \left(1 + \frac{v_1^2}{c^2}\right) \tag{4}$$

$$\Rightarrow \left(\frac{v_1}{c}\right)^2 - \frac{1}{\mathcal{M}_0}\left(1 + \mathcal{M}_0^2\right)\left(\frac{v_1}{c}\right) + 1 = 0 \tag{5}$$

This quadratic in v_1/c has the roots

$$\frac{v_1}{c} = \frac{1}{2}\left(\frac{1}{\mathcal{M}_0}(1 + \mathcal{M}_0^2) \pm \sqrt{\frac{1}{\mathcal{M}_0^2}(1 + \mathcal{M}_0^2)^2 - 4}\right) \tag{6}$$

$$= \frac{(\mathcal{M}_0^2 + 1) \pm \sqrt{(\mathcal{M}_0^2 - 1)^2}}{2\mathcal{M}_0} \tag{7}$$

$$= \mathcal{M}_0 \text{ or } \frac{1}{\mathcal{M}_0}. \tag{8}$$

These two roots are clearly the upstream and downstream velocities. Having no change, *i.e.* $v_1 = v_0$, also satisfies the jump conditions. However, we want the density and velocity to change across the shock, so $v_1 = c/\mathcal{M}_0$. Substituting this result into the mass jump condition, $\rho_1 = (v_0/v_1)\rho_0$, then yields the desired $\rho_1 = \mathcal{M}_0^2\rho_0$.

b) The isothermal equation of state gives the downstream gas pressure

$$P_1 = P_0(\rho_1/\rho_0) = \rho_1(P_0/\rho_0) = (\mathcal{M}_0^2\rho_0)c^2 = \rho_0 v_0^2. \tag{9}$$

The next-to-last step used the result for ρ_1 from part a). Thus, the downstream gas pressure is equal to the upstream ram pressure.

c) For a shock speed of 50 km s^{-1} and a sound speed of $c = 3$ km s^{-1}, the preshock Mach number is $\mathcal{M}_0 = 16.7$. Since the upstream and downstream mean molecular weights are the same, the equation relating the number densities n_0 and n_1 is the same as that relating ρ_0 and ρ_1. Thus, $n_1 = \mathcal{M}_0^2 n_0 = (16.7)^2(1 \text{ atom cm}^{-3}) = 280$ atoms cm^{-3}. Because of this compression, if the shock sweeps through a length ℓ of gas with a constant density, that gas will be compressed into a shell of width $\ell(n_0/n_1) = \ell/\mathcal{M}_0^2$ behind the shock. Or one could determine the thickness of the shell by realizing that material that flowed through the shock at the beginning of the period will have moved a distance $v_1 t$ from the shock in time t. A speed of 1 km s^{-1} is equal to 1.02 pc per 10^6 years. Thus a shock moving at 50 km s^{-1} covers a distance of 5.1 pc in 10^5 years. The material swept up in this time is compressed into a shell of thickness $5.1/280 = 0.018$ pc. Isothermal shocks produce shells that are thin compared to their diameters.

2. The interstellar medium is composed of a few "phases" with different number densities and temperatures. The different phases are in approximate pressure equilibrium, so all have the same value for nT. I chose the density and temperature used in this problem to match the value in the cool, neutral phase of the interstellar medium: 10 atoms cm^{-3} and 10^2 K. The average density for the interstellar medium is about 1 atom cm^{-3}, but there is not a lot of gas with that particular value.

a) The speed of sound for a perfect gas of neutral hydrogen with an adiabatic index, Γ, of 5/3 is

$$c_s = \left(\frac{\Gamma P}{\rho}\right)^{1/2} = \left(\frac{5}{3}\frac{kT}{\mu m_H}\right)^{1/2} \tag{10}$$

The mean molecular weight, μ, is 1.0 for pure *neutral* hydrogen. In class I noted that completely ionized gas with a solar composition has $\mu = 0.62$. Here we are interested in neutral gas, which for a solar composition has $1/\mu = (0.706)(1)+(0.275)(1/4)+(0.00959)(1/8)+\ldots \Rightarrow \mu = 1.3$. These different values for μ change the sound speed only slightly, so, for simplicity, I assumed pure hydrogen. Thus

$$c_s = \left(\frac{5}{3}\frac{(1.38\times10^{-16}\text{ erg K}^{-1})(10^2\text{ K})}{(1.0)(1.67\times10^{-24}\text{ g})}\right)^{1/2}\left(\frac{T}{10^2\text{ K}}\right)^{1/2} \tag{11}$$

$$= (1.2\times10^5\text{ cm s}^{-1})\left(\frac{T}{10^2\text{ K}}\right)^{1/2} \tag{12}$$

Thus, the speed of sound in the unperturbed interstellar medium with $T = 10^2$ K is 1.2 km s^{-1} (using $\mu = 1.3$ would yield 1.0 km s^{-1}).

b) The temperature behind a strong non-radiating shock with shock speed v_{sh} and, thus, Mach number $\mathcal{M}_0 = v_{sh}/c_s$ is

$$T_1 = \frac{(\Gamma - 1)(2\Gamma)}{(\Gamma + 1)^2}\mathcal{M}_0^2 T_0 = \frac{5}{16}\left(\frac{v_{sh}}{1.2 \text{ km s}^{-1}}\right)^2 (10^2 \text{ K}) = (31 \text{ K})\left(\frac{v_{sh}}{1.2 \text{ km s}^{-1}}\right)^2 \tag{13}$$

The table below contains $\mathcal{M}_0$ and T_1 for the three values of v_{sh}.

Table 1: Properties of a Strong Non-Radiating Shock

v_{sh} (km s^{-1})	$\mathcal{M}_0$	T_1 (K)	e (erg cm^{-3})	$\frac{de}{dt}$ (erg cm^{-3} s^{-1})	t_{cool} (yrs)
5.0×10^3	4.2×10^3	5.5×10^8	9.1×10^{-6}	9.0×10^{-20}	3.2×10^6
500	420	5.5×10^6	9.1×10^{-8}	2.0×10^{-20}	1.4×10^5
50	42	5.5×10^4	9.1×10^{-10}	3.2×10^{-19}	9.0×10^1

c) For the temperatures calculated in the previous part, the hydrogen will be ionized. The density behind a strong shock is 4× the upstream value. Thus, the electron and ion densities will be $n = 40$ cm^{-3} assuming a composition of pure hydrogen. The thermal energy density is then given by (note the unfortunate change in nomenclature from the book and lecture, where ρe is the energy density)

$$e = \frac{3}{2}(n_e + n_{ion})kT = (1.66 \times 10^{-10} \text{ erg cm}^{-3})\left(\frac{T}{10^4 \text{ K}}\right) \tag{14}$$

The resulting values are in the fourth column of the table above.

The Garmire notes gives the following equation for the power emitted by the thermal bremsstrahlung process per unit volume (see also equation 3.151 in the text)

$$j_{bremss}(T) = (2.4 \times 10^{-27} \text{ erg cm}^{-3} \text{ s}^{-1})\left(\frac{T}{1 \text{ K}}\right)^{1/2}\left(\frac{n_e}{1 \text{ cm}^{-3}}\right)^2. \tag{15}$$

This equation assumes "cosmic abundances" (*i.e.*, the abundances of elements found in the Sun). Pure hydrogen would yield a value about 1.4× smaller. I used the above equation to get slightly more realistic cooling times. The graph on the last page of the Garmire notes shows that bremsstrahlung emission is dominant for $T > 3 \times 10^7$ K. At lower temperatures recombination radiation and, especially, collisionally-excited line emission are more important. Thus, at $T = 5.5 \times 10^8$ K the energy loss rate per volume is $de/dt = j_{bremss}$. At $T = 5.5 \times 10^6$ K, all three emission mechanisms make significant contributions and I read the emitted power from the "total" line of the Garmire graph. Note that the value from the graph must be multiplied by our n_e^2 to obtain de/dt. Finally, at $T = 5.7 \times 10^4$ K, line emission by hydrogen produces most of the energy loss from the gas and I read the value from the plot handed out with the problem set. This value also needs to be multiplied by n_e^2. The resulting three values for de/dt are listed in the fifth column of the above table and the cooling timescale, $t_{cool} \equiv e/(de/dt)$, is in the sixth column.

d) The cooling time for a shock velocity of 5000 km s^{-1} is long compared to the typical $\sim 10^5$ yr lifetime of a supernova remnant. Thus, the assumption of a non-radiating shock is appropriate for this velocity. The cooling time is comparable to the age for a shock velocity of 500 km s^{-1} and the assumption is probably still OK (more detailed calculations are needed to be sure). A shock velocity of 50 km s^{-1} has a cooling time that is shorter than the lifetime of the remnant (and shorter than the time needed to reach that velocity assuming the evolution of the shock velocity given by theory for the Sedov phase — $\sim 3 \times 10^4$ yrs). Thus, at this velocity the shock is isothermal and the remnant is in the snowplow phase.

3. Cosmic rays moving in a magnetic field follow curving paths because of the Lorentz force. As discussed in class, this force is perpendicular to the momentum of the particle and so does no work. Thus, the energy and the magnitude of the momentum of the particle are constant. If the particle has rest mass m and charge Ze, then a particle moving perpendicular to the field lines follows a circular path with angular frequency

$$\omega = \frac{ZeB}{\gamma mc}. \tag{16}$$

Here $\gamma = 1/\sqrt{1-(v/c)^2}$. The Larmor radius of the orbit, r_L, is then given by

$$2\pi r_L = v(\text{period}) = \frac{2\pi v}{\omega} \Rightarrow r_L = v\left(\frac{\gamma mc}{ZeB}\right). \tag{17}$$

Cosmic rays are in the extreme relativistic limit and so $v = c$ and

$$\begin{aligned}
r_{L,rel} &= \frac{\gamma mc^2}{ZeB} = \frac{E}{ZeB} && (18)\\
&= \left(\frac{E}{1\text{ eV}}\right)\left(\frac{1\ \mu\text{G}}{B}\right)\left(\frac{(1\text{ eV})(1.602\times10^{-12}\text{ erg eV}^{-1})}{Z(4.803\times10^{-10}\text{ statcoloumb})(10^{-6}\text{ Gauss})}\right) && (19)\\
&= \left(\frac{3.3\times10^{3}\text{ cm}}{Z}\right)\left(\frac{E}{1\text{ eV}}\right)\left(\frac{1\ \mu\text{G}}{B}\right) && (20)\\
&= \left(\frac{1.1\times10^{-15}\text{ pc}}{Z}\right)\left(\frac{E}{1\text{ eV}}\right)\left(\frac{1\ \mu\text{G}}{B}\right) && (21)
\end{aligned}$$

a) Plugging $r_L = 10^4$ pc, $B = 1\ \mu$G, and $Z = 1$ into Equation 21 yields $E = 9.1 \times 10^{18}$ eV for the energy of a cosmic ray proton which has a Larmor radius equal to the radius of our Galaxy.

b) A cosmic ray proton with an energy of 8×10^{19} eV moving in a magnetic field of 1 μG has $r_L = 8.8 \times 10^4$ pc. If this proton moves 10 kpc through the Galaxy while experiencing that magnetic field, then the angular deflection can be estimated by taking the distance traveled to be the portion of the circular path traversed. This will be reasonably accurate as long as the path covered is much smaller than the radius of curvature. Then the angular deflection is

$$\theta = \left(\frac{10^4 \text{ pc}}{2\pi r_L}\right)(2\pi) = \frac{10^4 \text{ pc}}{(8.8 \times 10^4 \text{ pc})} = 0.11 \text{ rad} = 6.5^\circ. \tag{22}$$

This deflection is larger than the 3° found by the Auger Collaboration for the correlation between the directions to active galactic nuclei and the direction of arrival of cosmic rays. Our calculation is too simple because the magnetic field of the Galaxy is not uniform in direction and magnitude. But the magnetic field is likely large enough to complicate efforts to trace even very energetic cosmic rays back to their points of origin.

c) Plugging $E = 10^{14}$ eV, $B = 10$ μG, and $Z = 1$ into Equation 21 yields $r_L = 0.011$ pc. This is much smaller than the several-parsec size of supernova remnants. Specifically, with an angular diameter of 30 arcminutes and a distance of 2.2 kpc, the SN 1006 remnant has a diameter of 19 pc.

4. The individual images in the mosaic are claimed to be 1.6 arcmin on a side. These images are not quite square, but the expanded image in the mosaic is claimed to be 0.8 arcmin on a side and is about twice as tall. So assume that the images are 1.6 arcmin high. At a distance of 2000 pc to the Crab nebula, this angular size corresponds to a physical size of

$$s = d\theta = \left(\frac{1.6 \text{ arcmin}}{60 \text{ arcmin}/1^\circ}\right)\left(\frac{2\pi}{360^\circ}\right)(2000 \text{ pc}) = 0.93 \text{ pc} = 2.9 \times 10^{13} \text{ km}. \tag{23}$$

The seven Chandra images were taken 25 November 2000, 18 December 2000, 9 January 2001, 31 January 2001, 21 February 2001, 15 March 2001, and 6 April 2001. Thus, 132 days in total elapsed between the initial and final image. The innermost bright ring surrounding the pulsar in the X-ray images is relatively stationary and is thought to be the shock where the cold relativistic wind from the pulsar is converted to a more slowly moving synchrotron-emitting plasma. Just outside of this ring, arc-shaped wisps move outward with time. A particularly clear arc is the one above and to the right of the pulsar. I measured the displacement on a printout of the picture with a ruler and divided by the height of the image to get the angular displacement. More technologically savvy members of the class read the pictures into an image display tool to make their measurements. Dividing the distance moved by this feature (about 4.4 arcsec = 1.3×10^{12} km) by the time yields a speed of 0.38c. Features at larger distances from the pulsar tend to move more slowly, around 0.03c. Hester et al. (2002, ApJL, 577, L49) report speeds of 0.03c – 0.5c and discuss possible interpretations of the features seen.