Ph 442 Solutions for Problem Set 2

1. a) The time to cross the region from r_s to $r_s/2$ with a velocity v_g is

$$t = \frac{r_s/2}{v_g} = \frac{(150 \text{ km})/2}{5000 \text{ km}} = 0.015 \text{ s.} \tag{1}$$

One way to calculate the infall time from rest is to approximate the path with an extremely elliptical orbit with an apocenter at the starting radius and a very small pericenter. For a 1 $\mathrm{M}_\odot$ central mass, we know from Kepler's Third Law that the period and semi-major axis of an orbit are related by $P^2 = a^3$ when P is measured in years and a in astronomical units (1.5×10^{13} cm). The infall time is then $P/2$ when a is equal to half of the initial radius. Thus,

$$\begin{aligned} t_{infall} &= P/2 = a^{3/2}/2 = \frac{1}{2}\left(\frac{r_s/2}{1.5 \times 10^{13} \text{ cm}}\right)^{3/2} = \frac{1}{2}\left(\frac{(1.5 \times 10^{7} \text{ cm})/2}{1.5 \times 10^{13} \text{ cm}}\right)^{3/2} \\ &= 1.8 \times 10^{-10} \text{ yr} = 0.0056 \text{ s.} \end{aligned} \tag{2}$$

Another way to calculate the infall time is to use $1/\sqrt{G\rho}$, where ρ is 1 $\mathrm{M}_\odot$ divided by the volume within r_s. The infall time is shorter than the calculated time to move from r_s to $r_s/2$, so a constant velocity of 5000 km s^{-1} would not seem to be a good approximation. Numerical models do suggest a relatively constant infall velocity in the gain region, so pressure gradients are presumably important in resisting the force of gravity.

b) If the infall velocity is constant, then a thin shell of mass with width dr spends an equal amount of time in each radial interval dr that it passes through. The volume of each spherical shell is $4\pi r^2 dr$, so the density changes as r^{-2}. The mass in the gain region is

$$\begin{aligned} M_g &= \int_{r_s/2}^{r_s} 4\pi r^2 \rho_0 \left(\frac{r_s}{r}\right)^2 dr = 4\pi\rho_0 r_s^3 \int_{1/2}^{1} dx = 2\pi\rho_0 r_s^3 \\ &= 2\pi(10^9 \text{ g cm}^{-3})(1.5 \times 10^7 \text{ cm})^3 = 2.1 \times 10^{31} \text{ g} = 0.011 \text{ M}_\odot. \end{aligned} \tag{3}$$

Thus, the amount of mass in the gain region is a small fraction of the mass in the proto-neutron star and of the mass in the infalling envelope of the star. Each spherical shell with mass dm and radius r has a gravitational potential energy of $-GMdm/r$, where $M = 1$ $\mathrm{M}_\odot$. This assumes that the amount of mass outside of the proto-neutron star is negligible, which is a good approximation according to the mass calculation above. The total potential energy of the mass in the gain region is then

$$\begin{aligned} U_g &= \int_{r_s/2}^{r_s} 4\pi r^2 \rho_0 \left(\frac{r_s}{r}\right)^2 \left(-\frac{GM}{r}\right) dr = -4\pi GM\rho_0 r_s^2 \int_{1/2}^{1} \frac{dx}{x} = -4\pi \ln(2) GM\rho_0 r_s^2 \\ &= -4\pi \ln(2)(6.67 \times 10^{-8} \text{ dyne cm}^2 \text{ g}^{-2})(2.0 \times 10^{33} \text{ g})(10^9 \text{ g cm}^{-3})(1.5 \times 10^7 \text{ cm})^2 \\ &= -2.6 \times 10^{50} \text{ erg.} \end{aligned} \tag{4}$$

The potential energy depends strongly on r_s, which is why the increase in the radius of the gain region due to convection in two- and three-dimensional models makes an explosion more likely to happen.

c) The reduction in the outward-going intensity of neutrinos is given by equation (3.39) from the text:

$$dI = -n\sigma I dr \Rightarrow d\ln(I) = -n\sigma dr, \tag{5}$$

where σ is the neutrino cross-section per nucleon and n is the nucleon density. Now $n = \rho/m_p$. Plugging the expression for ρ into the right-hand side of equation 5 and integrating that side from $r_s/2$ to r_s and integrating the left-hand side from I_{in} to I_{out} yields

$$\begin{aligned} \ln\left(\frac{I_{out}}{I_{in}}\right) &= -\int_{r_s/2}^{r_s} \sigma\left(\frac{\rho_0}{m_p}\right)\left(\frac{r_s}{r}\right)^2 dr \\ &= -\left(\frac{\rho_0 r_s \sigma}{m_p}\right)\int_{1/2}^{1}\frac{dx}{x^2} = -\left(\frac{\rho_0 r_s \sigma}{m_p}\right) \end{aligned} \tag{6}$$

An intuitive way to write this result uses $\ell_0 \equiv m_p/(\rho_0\sigma)$, which is the neutrino mean free path at density ρ_0. If the fractional change in the neutrino intensity is small, then $\ln(I_{out}/I_{in}) = \ln((I_{in} - \Delta I)/I_{in}) \approx -\Delta I/I_{in}$. Then

$$\frac{\Delta I}{I_{in}} = \frac{r_s}{\ell_0}. \tag{7}$$

The neutrino cross section is

$$\sigma = \sigma_0\left(\frac{E_\nu}{m_e c^2}\right)^2 = (1.76\times10^{-44}\ \mathrm{cm}^2)\left(\frac{20\ \mathrm{MeV}}{0.511\ \mathrm{MeV}}\right)^2 = 2.7\times10^{-41}\ \mathrm{cm}^2 \tag{8}$$

and so

$$\ell_0 = \frac{1.67\times10^{-24}\ \mathrm{g}}{(10^9\ \mathrm{g\ cm}^{-3})(2.7\times10^{-41}\ \mathrm{cm}^2)} = 6.2\times10^7\ \mathrm{cm}. \tag{9}$$

Thus,

$$\frac{\Delta I}{I_{in}} = \frac{1.50\times10^7\ \mathrm{cm}}{6.2\times10^7\ \mathrm{cm}} = 0.24. \tag{10}$$

This fraction is not much less than one, but that approximation is comparable to (or, likely, better than) those made for other aspects of this problem.

d) If the neutrino luminosity is L, then in the time t that a bit of matter takes to fall through the gain region, it will absorb the energy

$$E = Lt(\Delta I/I) = (7\times10^{52}\ \mathrm{erg\ s}^{-1})(0.015\ \mathrm{s})(0.24) = 2.5\times10^{50}\ \mathrm{erg}. \tag{11}$$

In this calculation, the energy absorbed is slightly smaller than the gravitational binding energy found in b). However, the two numbers are of the same order, suggesting that the energy absorbed from neutrinos might be able to drive an explosion. In reality, matter in the gain region is also losing energy to neutrino emission and it is the difference between the energy gain and loss that must lead to the explosion. Actual explosions have been difficult to produce in the detailed numerical models.

2. Each neutrino leaving the neutron star carries a momentum $p_\nu = E_\nu/c$, since they are ultra-relativistic. It the neutrinos are emitted isotropically, they carry off no *net* momentum. However, if the emission is anisotropic, the neutrinos will have a net momentum and the conservation of momentum requires the the neutron star have an equal and opposite momentum.

Imagine taking 1% of the neutrinos and emitting them all in one direction. This will give the neutron star a velocity given by

$$\begin{aligned} M_{ns}v_{ns} &= (0.01)(E/c) && (12)\\ \Rightarrow v_{ns} &= (0.01)\left(\frac{E}{cM_{ns}}\right) = (0.01)\left(\frac{10^{51}\ \mathrm{erg}}{(3.0\times10^{10}\ \mathrm{cm\ s^{-1}})(1.4)(2.0\times10^{33}\ \mathrm{g})}\right) && (13)\\ &= 1.2\times10^{7}\ \mathrm{cm\ s^{-1}} = 120\ \mathrm{km\ s^{-1}}. && (14)\end{aligned}$$

Studies of the proper motions of pulsar show that they were formed with a typical velocity "kick" of about 400 km s^{-1} (see, for example, Hobbs *et al.*, 2005, MNRAS, 360, 974).

3. In the "laboratory" frame of the earth, the neutrinos have an energy E and a velocity v that is only very slightly less than c. Again in this frame, the time taken for the neutrinos to reach earth is $t = D/v$, where D is the distance. To find the velocity from the energy, use

$$E = \frac{m_0c^2}{(1-(v/c)^2)^{1/2}} \Rightarrow v = c\left(1-\left(\frac{m_0c^2}{E}\right)^2\right)^{1/2}. \quad (15)$$

Thus, the time to arrive at earth is

$$t = \frac{D}{c\left(1-\left(\frac{m_0c^2}{E}\right)^2\right)^{1/2}} \simeq \frac{D}{c}\left(1+\frac{1}{2}\left(\frac{m_0c^2}{E}\right)^2\right). \quad (16)$$

The last approximation is justified since $E \gg m_0c^2$. A massless neutrino travels at c and arrives in the time D/c. So the time delay with respect to a massless neutrino is

$$\begin{aligned}\Delta t &= \frac{D}{2c}\left(\frac{m_0c^2}{E}\right)^2 && (17)\\ &= \frac{(5.0\times10^4\ \mathrm{pc})(3.09\times10^{18}\ \mathrm{cm\ pc^{-1}})}{2(3.0\times10^{10}\ \mathrm{cm\ s^{-1}})}\left(\frac{10\ \mathrm{eV}}{10^7\ \mathrm{eV}}\right)^2\left(\frac{m_0c^2}{10\ \mathrm{eV}}\right)^2\left(\frac{10\ \mathrm{MeV}}{E}\right)^2 && (18)\\ &= (2.6\ \mathrm{s})\left(\frac{m_0c^2}{10\ \mathrm{eV}}\right)^2\left(\frac{10\ \mathrm{MeV}}{E}\right)^2. && (19)\end{aligned}$$

The figure handed out in class showed that neutrinos from SN1987A were observed by the Kamiokande and IMB (Irvine-Michigan-Brookhaven) water Cerenkov detectors over a time interval of about 12.3 s. The typical energy of the neutrinos is a bit trickier to estimate because of the thresholds in energy for the detection of neutrinos. The IMB detector has a larger volume, but a high detection threshold of about 20 MeV. Kamiokande had a threshold of 5 MeV and using just the neutrinos observed by that detector suggests a mean energy of about 10 MeV. Substituting these values for Δt and E in equation 19 argues that the rest mass of the neutrino cannot be much larger than 22 eV. If the observed spread of arrival times were due to differing energies, then the the most energetic neutrinos should have arrived first. This does not seem to be the case because, while neutrinos with high energy did tend to arrive early in the burst, neutrinos with lower energy arrived both early and late. The data looks more like a decrease in the mean energy of the emitted neutrinos with time, which is expected from models of core-collapse supernovae. A statistical analysis of neutrino arrival time *versus* energy showed that an electron neutrino mass larger than about 15 eV could be ruled out with 95% confidence, which was better than the laboratory limits from double beta decay at the time. Upper limits on the mass of the electron neutrino from laboratory measurements are currently about 2 eV and the limit from cosmological measurements of the matter density of the universe are that the sum of the rest masses of all three neutrino types must be less than 0.67 eV.

4. a) A necessary condition for the white dwarf to be disrupted (*i.e.*, its mass dispersed to infinity) is that the energy released by the nuclear burning exceed the gravitational binding energy (*i.e.*, the gravitational potential energy). The energy remaining after dispersing the star can go into kinetic energy, which yields the average velocity of the ejecta. A ^{12}C nucleus has a binding energy per nucleon of 7.68 MeV, while that of ^{56}Fe is 8.79 MeV. Thus, burning ^{12}C to ^{56}Fe releases 1.11 MeV per nucleon. This is close to the 1 MeV per nucleon $= 9.7 \times 10^{17}$ erg g^{-1} given in the text. (Real white dwarfs also contain ^{16}O with a binding energy of 7.98 MeV per nucleon.) Using the value from the text, burning 0.7 M$_\odot$ $= 1.4 \times 10^{33}$ g of material releases 1.4×10^{51} erg of nuclear energy. The other 0.7 M$_\odot$ of the white dwarf burns to intermediate-mass elements and produces nearly as much energy as burning to iron. The energy released by burning the entire 1.4 M$_\odot$ to ^{56}Fe would release 2.8×10^{51} erg.

The gravitational binding energy is dimensionally $E_g = -GM^2/R$, where M is the mass and R is the radius of the star. (As an aside, this simple formula is quite good. A constant-density sphere has $E_g = -0.6GM^2/R$, but real stars are centrally concentrated and are more bound. For a white dwarf whose interior obeys the equation of state for non-relativistic degenerate electrons, $P \propto \rho^{5/3}$, the equations of stellar structure can be solved analytically to show that $E_g = -0.86GM^2/R$. Similarly, for relativistic degeneracy, $E_g = -1.5GM^2/R$.) For $M = 1.4$ M$_\odot$ $= 2.8 \times 10^{33}$ g and $R = 2 \times 10^6$ m $= 2 \times 10^8$ cm, the simple expression gives $E_g = -2.7 \times 10^{51}$ erg.

Thus, most of the white dwarf must be burned (nearly) to iron to produce enough energy to unbind the star. Taking the largest nuclear energy release, the residual kinetic energy of the debris is 0.1×10^{51} erg. This value implies an average velocity of

$$v = \sqrt{\frac{2\Delta E}{M}} = \sqrt{\frac{2(0.1 \times 10^{51}\ \mathrm{erg})}{2.8 \times 10^{33}\ \mathrm{g}}} = 3 \times 10^8\ \mathrm{cm\ s^{-1}} = 3 \times 10^3\ \mathrm{km\ s^{-1}}. \tag{20}$$

This is the same order of magnitude as the observed expansion velocities of supernova remnants.

b) The decay of ^{56}Ni to ^{56}Co releases 3.0×10^{16} erg g^{-1} and the decay of ^{56}Co to ^{56}Fe releases 6.4×10^{16} erg g^{-1}. Thus, the decay of 0.7 M$_\odot$ of ^{56}Ni to ^{56}Fe releases $(1.4 \times 10^{33}\ \mathrm{g})(9.4 \times 10^{16}\ \mathrm{erg\ g^{-1}}) = 1.3 \times 10^{50}$ erg. Thus, the energy released in the decays, which power the visible supernova, is only 0.05 times the maximum amount of energy released in the initial nuclear explosion.