Ph 442 Solutions for Problem Set 1

1. This question addresses whether the universe is transparent enough that astronomy is possible at X-ray wavelengths. The analytic expression for the photoelectric cross-section of the interstellar medium given in the Garmire notes is

$$\sigma_{bf}(E) = \left(2 \times 10^{-22}\text{cm}^2\ (\text{H atom})^{-1}\right)\left(\frac{E}{1\ \text{keV}}\right)^{-8/3}, \tag{1}$$

where E is the photon energy. The mean free path for a photon through a medium of hydrogen atoms with number density n_H and cross-section σ_{bf} is $\ell = 1/(n_H \sigma_{bf})$. Plugging in numbers yields

$$\begin{aligned} \ell &= \frac{1}{(1.0\ (\text{H atom})\ \text{cm}^{-3})(2 \times 10^{-22}\ \text{cm}^2\ (\text{H atom})^{-1})(E/(1\ \text{keV}))^{-8/3}} && (2) \\ &= \left(5.0 \times 10^{21}\ \text{cm}\right)\left(\frac{E}{1\ \text{keV}}\right)^{8/3} && (3) \\ &= \left(1.6 \times 10^{3}\ \text{pc}\right)\left(\frac{E}{1\ \text{keV}}\right)^{8/3}. && (4) \end{aligned}$$

The column density of hydrogen atoms corresponding to the mean free path is $N_H = n_H \times \ell = n_H \times (1/(n_H \sigma_{bf})) = 1/\sigma_{bf}$.

The cross-sections, mean-free-paths, and column densities are then

a. $E = 20$ eV: $\sigma_{bf} = 6.8 \times 10^{-18}$ cm^2, $\ell = 0.047$ pc, and $N_H = 1.5 \times 10^{17}$ (H atoms) cm^{-2}.

b. $E = 200$ eV: $\sigma_{bf} = 1.5 \times 10^{-20}$ cm^2, $\ell = 22$ pc, and $N_H = 6.8 \times 10^{19}$ (H atoms) cm^{-2}.

c. $E = 2$ keV: $\sigma_{bf} = 3.1 \times 10^{-23}$ cm^2, $\ell = 1.0 \times 10^4$ pc, and $N_H = 3.2 \times 10^{22}$ (H atoms) cm^{-2}.

d. $E = 20$ keV: $\sigma_{bf} = 6.8 \times 10^{-26}$ cm^2, $\ell = 4.7 \times 10^6$ pc, and $N_H = 1.5 \times 10^{25}$ (H atoms) cm^{-2}.

2. The intergalactic medium is not uniform — it is likely formed into sheets and filaments — but an average density will yield a good approximation to the absorption for path-lengths larger than a few tens of megaparsecs. The best estimates of the average electron density are probably those coming from the effect of electron scattering on the anisotropies of the cosmological microwave background radiation. The Thompson scattering cross-section is $\sigma_T = (8\pi/3)r_0^2$, where $r_0 \equiv e^2/(m_e c^2) = 2.82 \times 10^{-13}$ cm is the classical electron radius. Plugging numbers into the formula gives $\sigma_T = 6.66 \times 10^{-25}$ cm^2. Then an average electron density of $n = 2.0 \times 10^{-7}$ cm^{-3} yields the mean-free-path:

$$\ell = \frac{1}{n\sigma_T} = \frac{1}{(2.0\ \times 10^{-7}\ \text{cm}^{-3})(6.66 \times 10^{-25}\ \text{cm}^2)} = 7.5 \times 10^{30}\ \text{cm} = 2.4 \times 10^{6}\ \text{Mpc}. \tag{5}$$

The radius of the visible universe, *i.e.*, the distance a photon can travel since the Big Bang, is about the speed of light multiplied by the time since the Big Bang. Since the age of the Universe is 13.5 Gyrs, the radius is 4300 Mpc (including the expansion of the universe would increase this by a factor of about 3). Thus, the intergalactic medium is transparent to X-rays.

A more careful treatment of this problem needs to include the increasing density of the universe at large look-back times (due to viewing the universe when it was less expanded). The mean free path needs to be reduced by a factor of $(2.4\times10^6\ \text{Mpc})/(4300\ \text{Mpc}) = 5.6\times10^2$, so the density would have to increase by that factor. Since the density varies as the cube of the "size" of the universe, this argues that electron scattering will not become important until we look to redshifts, z, greater than 7.

3. The ^{56}Co nucleus decays to ^{56}Ni with a half-life of $\tau_{1/2} = 77.7$ days and releases 6.4×10^{16} erg for each gram of ^{56}Co that decays.

a) If all of the $(0.075\ \text{M}_\odot)(2.0\times10^{33}\ \text{g M}_\odot^{-1}) = 1.5\times10^{32}$ g of ^{56}Co decays, it will release $E_0 = 9.6\times10^{48}$ erg. The amount of energy released per second is proportional to the rate at which nuclei decay. Thus

$$\frac{dE}{dt} \propto \frac{dN}{dt} = \frac{d}{dt}\left(N_0 e^{-\ln(2)t/\tau_{1/2}}\right) = -\frac{\ln(2)}{\tau_{1/2}}\left(N_0 e^{-\ln(2)t/\tau_{1/2}}\right). \tag{6}$$

Since the decay of all of the nuclei releases E_0, clearly

$$\frac{dE}{dt} = \frac{\ln(2)E_0}{\tau_{1/2}}\left(e^{-\ln(2)t/\tau_{1/2}}\right) = (9.9\times10^{41}\ \text{erg s}^{-1})e^{-t/(112\ \text{days})}. \tag{7}$$

Just after the supernova ($t = 0$), the rate of energy release is 9.9×10^{41} erg s^{-1} $= 2.6\times10^8$ L$_\odot$ and after 1 yr ($t = 365.25$ days), the rate is 3.8×10^{40} erg s^{-1} $= 9.9\times10^6$ L$_\odot$.

b) There are at least two approaches to solving this problem. One is to assume that each decay yielded an average energy of 847 keV + 0.68(1238 keV) and to use this to convert the energy release rate from part (a) into the rate at which 847 keV gamma-rays are produced. This approach has the problem that energy is released in other ways by the decay, but is actually reasonably accurate. A simpler approach is to use the mass of the ^{56}Co nucleus, 55.935 atomic mass units, to calculate that SN1987A started with $N_0 = (1.5\times10^{32}\ \text{g})/((55.935)(1.66\times10^{-24}\ \text{g})) = 1.6\times10^{54}$ ^{56}Co nuclei. Then the rate at which 847 keV gamma-rays are produced is

$$L_{847} = -\frac{dN}{dt} = \frac{N_0\ln(2)}{\tau_{1/2}}e^{-\ln(2)t/\tau_{1/2}} = (1.7\times10^{47}\text{s}^{-1})e^{-t/(112\ \text{days})}. \tag{8}$$

At $t = 0.5$ yr $= 183$ days, $L_{847} = 3.3\times10^{46}$ s. Finally, the flux of gamma-rays observed at Earth is

$$F_{847} = \frac{L_{847}}{4\pi d^2} = \frac{3.3\times10^{46}\ \text{s}}{4\pi((5.0\times10^4\ \text{pc})(3.09\times10^{18}\ \text{cm pc}^{-1}))^2} = 0.11\ \text{cm}^{-2}\ \text{s}^{-1}. \tag{9}$$

c) The observed rate of 1.0×10^{-3} 847 keV photons $\mathrm{cm}^{-2}\ \mathrm{s}^{-1}$ is a factor of 100 smaller than the rate calculated in (b). The reason is that most of the photons are absorbed in the supernova ejecta. Indeed, it is this thermalized energy that produces the exponential decline of the luminosity of the supernova in visible-light. Models of supernova that do not include any mixing during the explosion actually predict that even fewer gamma-rays would be observed. Material from the deep interior of the progenitor star must have been mixed closer to the surface during the explosion so that the decay photons were able to escape. This is evidence for the vigorous convection and large departures from spherical symmetry that we believe are necessary to revive the stalled shock and produce a supernova explosion.

4. a) For a mean neutrino energy of 10 MeV, we can ignore the rest-mass energy of the neutrinos (see part c). Thus, the number of neutrinos produced by SN1987A was

$$(10^{53}\ \mathrm{erg})/((10^{7}\mathrm{eV})(1.6 \times 10^{-12}\ \mathrm{erg\ eV}^{-1})) = 6.3 \times 10^{57}. \tag{10}$$

The total number of neutrinos passing through a unit area at the Earth, called the *fluence*, is

$$\mathcal{F} = \frac{6.3 \times 10^{57}}{4\pi((5.0 \times 10^{4}\ \mathrm{pc})(3.09 \times 10^{18}\ \mathrm{cm\ pc}^{-1}))^2} = 2.1 \times 10^{10}\ \mathrm{cm}^{-2}. \tag{11}$$

b) The mean free path for neutrinos scattering from nucleons is $\ell = 1/(n\sigma)$, where n is the nucleon density and σ is the scattering cross section given by

$$\sigma_\nu = \sigma_0 \left(\frac{E_\nu}{m_e c^2}\right)^2, \tag{12}$$

and

$$\sigma_0 = \frac{4G_F^2 m_e^2}{\hbar^4} = 1.76 \times 10^{-44}\ \mathrm{cm}^2. \tag{13}$$

To a good approximation, the nucleon density is the matter density, ρ, divided by the mass of a proton (or neutron), m_p. Subsitituting this and equation 12 into the expression for ℓ yields

$$\begin{aligned}
\ell_\nu &= \frac{1}{(\rho/m_p)\sigma_0(E_\nu/(m_e c^2))^2} && (14)\\
&= \frac{1}{\left(\frac{10^{12}\ \mathrm{g\ cm}^{-3}}{1.67\times10^{-24}\ \mathrm{g}}\right)\left(\frac{\rho}{10^{12}\ \mathrm{g\ cm}^{-3}}\right)(1.76 \times 10^{-44}\ \mathrm{cm}^2)\left(\frac{10^4\ \mathrm{keV}}{511\ \mathrm{keV}}\right)^2\left(\frac{E_\nu}{10\ \mathrm{MeV}}\right)^2} && (15)\\
&= (2.5 \times 10^5\ \mathrm{cm})\left(\frac{E_\nu}{10\ \mathrm{MeV}}\right)^{-2}\rho_{12}^{-1}. && (16)
\end{aligned}$$

The difference between the 2.5×10^5 cm in the above equation and the 2.0×10^5 cm is small.

c) A uniform-density sphere with mass M has $\rho = M/(4\pi R^3/3) \Rightarrow R^2 = (3M/(4\pi\rho))^{2/3}$. Substituting this expression and that for ℓ_ν into equation

$$t_{diff} = \left(\frac{\ell_\nu}{c}\right)\left(\frac{R^2}{\ell_\nu^2}\right) = \frac{R^2}{\ell_\nu c}. \tag{17}$$

yields

$$t_{diff} = \left(\frac{3M}{4\pi\rho}\right)^{2/3} \left(\frac{1}{c(2.5\times 10^5\ \mathrm{cm})}\right) \left(\frac{E_\nu}{10\ \mathrm{MeV}}\right)^2 \left(\frac{\rho}{10^{12}\ \mathrm{g\ cm^{-3}}}\right) \tag{18}$$

$$= \left(\frac{3(1.4\ \mathrm{M_\odot})(2.0\times 10^{33}\ \mathrm{g\ M_\odot^{-1}})}{4\pi(10^{12}\ \mathrm{g\ cm^{-3}})}\right)^{2/3} \left(\frac{(E_\nu/10\ \mathrm{MeV})^2(\rho/10^{12}\ \mathrm{g\ cm^{-3}})^{1/3}}{(3.0\times 10^{10}\ \mathrm{cm\ s^{-1}})(2.5\times 10^5\ \mathrm{cm})}\right) \tag{19}$$

$$= (1.0\times 10^{-2}\ \mathrm{s}) \left(\frac{E_\nu}{10\ \mathrm{MeV}}\right)^2 \left(\frac{\rho}{10^{12}\ \mathrm{g\ cm^{-3}}}\right)^{1/3}. \tag{20}$$

For a density of $\rho = 10^{15}$ g cm^{-3}, the diffusion time is

$$t_{diff} = (0.10\ \mathrm{s}) \left(\frac{E_\nu}{10\ \mathrm{MeV}}\right)^2. \tag{21}$$

This seems a little short as the neutrinos observed from SN1987A had a typical energy of about 10 Mev and arrived over a time of about 10 s. Still, our estimate is not bad for such a simple calculation.