Ph 441/541 Problem Set 5

Due: Friday, March 2, 2012

1. Saha Equation:
The text explored the Saha equation using pure hydrogen gas as an example. Now consider the more complicated ^{4}He atom with its two electrons and work out the coupled pair of equations describing ionization in a pure helium gas. Assume that the neutral atom and first ionized ion are in their respective ground states (thus, we ignore the complication of partition functions in this problem). The ionization potentials to remove the first (second) electron are $E_1 = 24.587$ eV ($E_2 = 54.416$ eV). To agree on a common nomenclature, let n_e, n_0, n_1, and n_2 be the number densities of, respectively, electrons, neutral atoms, and once and twice ionized ions. The total number density of atoms plus ions of the pure helium gas is denoted by n. Furthermore, define z_e as the ratio n_e/n and, in a like manner, let z_i be n_i/n, where $i = 0, 1, 2$. The gas is assumed to be electrically neutral. The degeneracy factors for the ground states of neutral, once ionized, and twice ionized helium are 1, 2, and 1, respectively.

a. Construct the ratios of $n_e n_1/n_0$ and $n_e n_2/n_1$. In doing so you must take care in establishing the zero points of energy for the various constituents. The final forms that you obtain should not contain any chemical potentials (and you must show why this is true).

b. Applying $n = n_0 + n_1 + n_2$ and overall charge neutrality, recast the above Saha equations so that only z_1 and z_2 appear as unknowns. The resulting two equations can have temperature and n or, equivalently, $\rho = 4m_A n$ as independent parameters (m_A is the atomic mass unit).

c. (Ph 541 students only) Simultaneously solve the two Saha equations for z_1 and z_2 for temperatures in the range 4×10^4 K $\leq T \leq 2 \times 10^5$ K with a fixed value of the density from among the choices $\rho = 10^{-1}$, 10^{-3}, or 10^{-5} kg m^{-3}. Choose a dense grid in temperature and plot your results. Once you have found z_1 and z_2, also calculate and plot z_e and z_0 for the same range of temperature. Note that this is a numerical exercise and the use of a computer is strongly advised.

2. Eddington Luminosity:
Show that heat transfer by radiative diffusion implies a non-zero gradient for the radiation pressure which is proportional to the flux of energy carried by the radiation. Bearing in mind that the magnitude of the force per unit volume in a fluid due to the pressure is equal to the pressure gradient, find the radiative energy flux which can, by itself, support the surface layers (atmosphere) of a star with surface gravity $g = G\mathcal{M}/\mathcal{R}^2$. Hence show that a star of mass $\mathcal{M}$ has a maximum luminosity (to avoid ejecting its atmosphere) given by

$$\mathcal{L}_{max} = 4\pi c G\mathcal{M}/\kappa,$$

where κ is the opacity near the surface. Obtain a numerical estimate for this luminosity in $\mathcal{L}_\odot$ as a function of $(\mathcal{M}/\mathcal{M}_\odot)$ by assuming that the surface is hot enough for the opacity to be principally provided by electron scattering.

3. White Dwarf Envelopes (Ph 541 students only):
Use Phillips Eq. (3.37) and show that the radiative temperature gradient in the outer envelope

of classical gas surrounding a white dwarf is given by

$$\frac{dT}{dr} = -\frac{G\mathcal{M}\overline{m}}{4.25r^2k}.$$

Estimate the thickness of the outer envelope of a white dwarf with mass $\mathcal{M} = 0.4\mathcal{M}_\odot$, radius $\mathcal{R} = \mathcal{R}_\odot/100$, and an internal temperature of 10^7 K.