Ph 441/541 Problem Set 1

Due: Friday, February 3, 2012

1. Hydrostatic Equilibrium:
Consider a sphere of mass M and radius R. Calculate the gravitational potential energy of the sphere assuming (a) a density which is independent of the distance from the center and (b) a density which increases towards the cener according to

$$\rho(r) = \rho_c(1 - r/R).$$

In both cases (a) and (b), also write down the average internal pressure needed for hydrostatic equilibrium and determine how the pressure within the sphere depends on the distance from the center.

2. Virial Theorem (Ph 441 students only):
As the Sun evolved towards the main sequence, it contracted due to gravity while remaining close to hydrostatic equilibrium and its internal temperature changed from about 30,000 K, Phillips Eq. (1.23), to about 6×10^6 K, Eq. (1.31). This stage of evolution is called the Kelvin-Helmholtz stage. Find the total energy radiated during this contraction. Assume that the luminosity during this contraction is comparable to the present luminosity of the Sun and estimate the time taken to reach the main sequence.

3. Schönberg-Chandrasehkar Limit (Ph 541 students only):
When nuclear burning ceases in the core of a star, the flux of energy flowing through the core and, hence, the temperature gradient in the core, become zero. The core becomes isothermal and the pressure gradient needed to support the core and the overlying envelope arises solely from a steep density gradient. However, if the isothermal core becomes too massive, it becomes unstable. It will contract and the overlying envelope will expand. This limiting value for the mass of an isothermal core is call the Schönberg-Chandrasekhar limit (Schönberg & Chandrasekhar 1942, ApJ, 155, 183). This limit is on the order of one-tenth the total mass of the star. However, the core mass can be a higher fraction if electron degeneracy becomes important in the core; this is more likely in less massive stars where the core has a higher density.

In this problem we consider the physics of the Schönberg-Chandrasekhar limit. Consider a star with mass M and radius R with an isothermal core of mass M_c, radius R_c, and volume V_c. Let T_c denote the uniform temperature of the core and let P_c denote the pressure exerted on the core by the envelope. Phillips section 1.2 derived Eq. (1.7), the virial theorem for the star as a whole, by multiplying the equation of hydrostatic equilibrium (1.5) by $4\pi r^3$ and integrating from the center of the star to the surface, where the pressure is zero.

a. This time integrate from the center to the edge of the isothermal core at $r = R_c$, where the pressure is P_c, and show that in hydrostatic equilibrium

$$3V_cP_c - 3\langle P\rangle_c = E_{GC},$$

where $\langle P\rangle_c$ is the average pressure in the core and E_{GC} is the potential energy of the core.

b. Assume that the core consists of an ideal gas of non-relativistic particles and show that

$$3V_cP_c = 2E_{KC} + E_{GC},$$

where E_{KC} is the translational kinetic energy of the gas particles in the core.

c. Now assume that the core consists of an ideal classical gas of particles with average mass m_c and show that

$$P_c = \frac{A}{R_c^3} - \frac{B}{R_c^4},$$

where A and B are positive constants.

4. Schönberg-Chandrasehkar Limit continued (Ph 541 students only):

a. Sketch the P_c from the last part of the previous problem as a function of R_c and show that it has a maximum value

$$P_0 = C\frac{(kT_c)^4}{G^3 m_c^4}\frac{1}{M_c^2}$$

at

$$R_0 = D\frac{GM_c m_c}{kT_c},$$

where C and D are numerical constants of order unity.

b. Use your sketch to show that P_0 is the maximum pressure that the core can withstand and still remain in hydrostatic equilibrium. You can do this by showing that, if the radius of the core is larger than R_0, then any small increase in the pressure on the core is accompanied by a contraction of the core, so the core is in a state of stable equilibrium. But if the radius of the core becomes smaller than R_0, then the core is at best in a state of unstable equilibrium, where any decrease in the radius is accompanied by a decrease in the pressure on the core.

c. Now note that the values of P_c and T_c depend on the total mass M and radius R of the star. In analogy with Phillips Eqs. (1.29) and (1.31)

$$P_c \propto \frac{GM^2}{R^4} \quad \text{and} \quad T_c \propto \frac{GMm}{R},$$

where m is the average mass of the gas particles in the star. Show that when the following condition is satisfied, P_c is less than the maximum pressure that it can sustain:

$$\frac{M_c}{M} < \alpha\left[\frac{m}{m_c}\right]^2,$$

where α is another numerical constant.

The theoretical value of the constant α is about 0.4. This imples that an isothermal core with $m_c = 2m$ has a maximum mass of about $0.1M$.