LEAST SQUARE FITTING

PURPOSE

   To learn the techniques involved in least square fitting a straight line to experimental data and determining the errors in the fitting parameters.
   To determine the lifetime of the metastable excited state of ^{137}Ba.

HOMEWORK

Read John R. Taylor "An Introduction to Error Analysis", Chapter 8.
Do problems 8.9, 8.25, 8.26.

EXPERIMENT

   The radioactive decay diagram of ^{137}Cs is shown below. The experiment you will be doing today, measuring the lifetime of the metastable state of ^{137}Ba (half-life 2.55 minutes) excited during the decay of ^{137}Cs, may be familiar to you since you may have already made this measurement in your introductory physics laboratory.
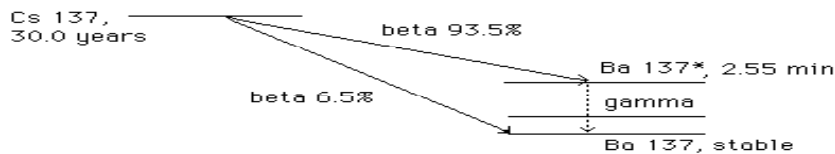
Here the emphasis will be on the fitting process one uses to extract the half life from the data. You will use the same apparatus as for the Distribution Functions experiment - a Geiger tube, counter, interface, and a computer - and will use the program LOGGER PRO to record the data. You will use MATLAB to plot the number of counts/interval D against the elapsed time t. This will give a characteristic exponential decay,

$$D = D_o \, e^{-t/\tau} \, ,$$

where τ is the mean lifetime of ^{137}Ba and D_o is the initial decay rate. The half-life is given by $\tau_{1/2} = \tau \ln 2$ (why?). When the data are replotted as the natural log of D vs. time, a straight line

$$\ln D = \ln D_o - t/\tau = A + Bt$$

will be obtained. Thus by performing a linear least squares fit as described below, you will obtain the coefficients A and B, and you will be able to calculate $\tau_{1/2}$ and D_o and their errors. You might be tempted to just use a linear fitting routine provided in another program – however, this lab is concerned with understanding how the fit is done so do not use a canned fit in this lab. Also, it is not often that such a canned fit program will handle the uncertainties of the data points correctly. (Most fitting programs just assume the uncertainties to be the same for all data points.)   The least square fitting process is discussed in the textbook and was explained in the lecture. In the next section we summarize the results using Taylor's notation.

BACKGROUND
        In fitting a set of data $(y_1 ,t_1), ... (y_N ,t_N)$ (where $y_1 = ln D_1$ etc.) each y_i is assumed to have uncertainty σ_i and the straight line you want to fit is assumed to give the "true" value Y_i for each t_i. The fitting process involves choosing the best fit line, $Y_i = A + Bt_i$, so that the overall discrepancy is minimized. Specifically the quantity

$$\chi^2 = \sum \left(\frac{(y_i - Y_i)^2}{\sigma_i^2} \right) \tag{1}$$

is minimized, hence the name least squares or least χ^2 fit. All sums Σ are understood to run from 1 to N. To minimize χ^2, Eq..(1) is differentiated with respect to A and to B and the results are individually set equal to zero and solved for A and B.

$$A = (M_{22}V_1 - M_{12}V_2)/\Delta \tag{2a}$$
$$B = (M_{11}V_2 - M_{21}V_1)/\Delta \tag{2b}$$
$$\Delta = M_{11}M_{22} - M_{12}M_{21} \tag{2c}$$

where

$$M11 = \sum \frac{1}{\sigma_i^2} \quad M12 = \sum \frac{t_i}{\sigma_i^2} \quad V1 = \sum \frac{y_i}{\sigma_i^2}$$

$$M21 = \sum \frac{t_i}{\sigma_i^2} \quad M22 = \sum \frac{t_i^2}{\sigma_i^2} \quad V2 = \sum \frac{t_i y_i}{\sigma_i^2} \qquad (2d)$$

As you know, in a counting experiment the error of a data point with value
N is given by $\sqrt{N}$, since the expected distribution is a Poisson distribution.
When plotted on a logarithmic scale, however, the length of the error bars
are modified. Use the error propagation formula to derive the uncertainties
σ_i of log(D_i).

The uncertainty in A and B is then found from

$$\sigma_A^2 = \frac{M_{22}}{\Delta} \qquad (3)$$

$$\sigma_B^2 = \frac{M_{11}}{\Delta} \qquad (4)$$

When calculating the errors in τ and D_O from A and B, the usual rules for
the propagation of error apply. Show all such work.

EQUIPMENT
Radioactive source (5 drops of solution containing meta-stable ^{137}Ba eluted
from a ^{137}Cs source)
Eluting apparatus
Electronic Counter
LabPro Interface
Computer

PROCEDURE
A. The Geiger tube should be connected to the BNC connector on the
back of the counter, and the DIG/SONIC1 input of the LabPro interface to the
jack on the back of the counter. The LabPro interface is connected to the
serial port of the computer.
B. Configure LOGGER PRO for a 600 second (10 minute) run with 10 s
intervals. This run will yield 60 data points for D*, the number of
decays/interval including background radiation. Set the graph to display

counts/interval vs. time. To ensure that no new ^{137}Ba nuclei will be created by ^{137}Cs decay in your sample while you are counting, your instructor will separate the Ba from the Cs by eluting with a weak EDTA solution into your sample holder. This must be done shortly before you take your decay data since the half life is fairly short and the activity is quite weak. About five drops of solution will give reasonable counting rates. Immediately after your sample is eluted, begin the experiment. After the 10 minute run, export the data to a file for import into MATLAB (see hints in previous week).

C. To account for the background radiation from your data, do not move the sample holder and wait about 20-30 minutes. Then repeat step B. This will give you the background radiation. Record D_b, the average number of counts per interval, and calculate the standard deviation in D_b from the LOGGER PRO display. It is important that for both steps B and C that the ^{137}Cs source be nowhere near your counter.

D. Import your table of 60 points (events/interval vs. time) into MATLAB. Calculate $D = D^* - D_b$, which is the true decay rate of your sample. ~~Note that $t_i = s_i - \Delta t/2$ where $\Delta t = interval\ size.$~~ Calculate also $y = log(D)$ and the uncertainties in D and y. For the uncertainty in D, if D_b, is zero, we use $\sqrt{D_i}$, and for $\ln D_i$, $\sigma_i = \dfrac{1}{\sqrt{D_i}}$. Derive this latter result and find also what is the uncertainty to use when D_b is not zero. Show the derivations in your lab report.

E. Plot $D = D^* - D_b$ vs. time t and $\ln D$ vs. t, and make a least square fit to a straight line for the latter curve. Do not use any prepackaged fitting routines, but use the formulas we derived, to properly account for the non-constant errors of the data points. Include error bars in the plots (use MATLAB errorbar() function). Superimpose the fitted curve on both plots. Print the graphs for your report.

F. Use Eqs.(3) and (4) to calculate σ_A and σ_B . Calculate χ^2 from equation (1b).

G. From $A, B,$ σ_A and σ_B calculate $\tau_{1/2}$ and D_O and their errors.

H. Your lab report will consist of your graphs and your (detailed) calculations of σ_A and $\sigma_B,$ and $\tau_{1/2}$ and D_O including their errors. Include your MATLAB m-file. Give a concise description of the experiment and the fitting technique, and discuss the determination of the half life and compare it to the accepted value. Be sure to state all significant results in the text, or

refer to clearly labeled and presented tables.