FORCED HARMONIC MOTION

PURPOSE

To study the resonant response of a system of a weight suspended from a spring where the system is driven up and down harmonically while subject to damping forces, and to determine both the relationships among frequency, amplitude and phase, and the effects of damping.

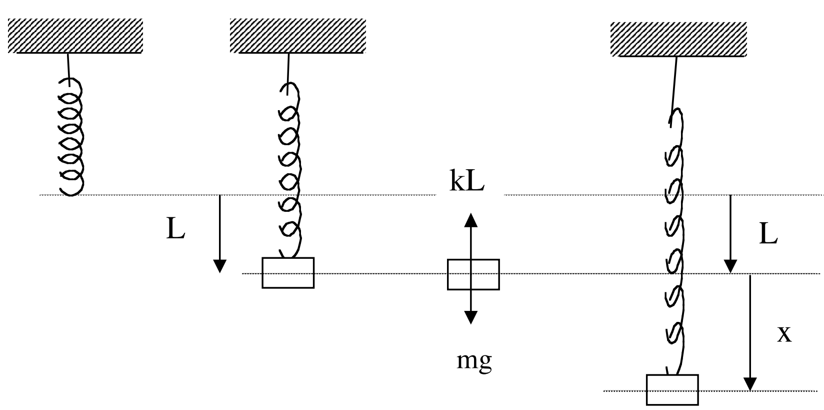
READINGS

Marion and Thornton, "Classical Dynamics", Chapter 3; or other Physics 381 dynamics textbook.

THEORY

Consider a spring with force constant k. When a weight mg is hung from the spring, it stretches by an amount L When the weight is motionless, we know by Newton's second law, $\Sigma F = ma$ and Hooke's law, $F = -kx$ that $mg - kL = 0$. When the spring is displaced a small distance x_0 from L and released, the weight undergoes periodic vertical vibrations in x about the equilibrium length L described by Newton's second law,

$$-k(L + x) + mg = ma$$



Using $kL = mg$ from above, we find

$$-kx = m\frac{d^2x}{dt^2} \qquad (1)$$

If the weight is released at t = 0, the solution is

$$x = x_0\cos(\omega_0 t) \qquad (2)$$

Where $\omega_0 = \sqrt{k/m}$. ω_0 is the natural or resonant frequency of the system and is expressed in radians/s. [Check that Eq.(2) is the solution by substituting it into Eq.(1).] Note that at t=0, x=0 and v=dx/dt=0.

Equation (2) tells us that the weight will continue to oscillate forever. This is because the system is assumed not to lose its starting energy. The term "damping" describes the condition where energy is lost. Damping forces act antiparallel to the velocity and are frequently found to obey Stoke's law, $F = -Rv = -Rdx/dt$, where R is a constant. The equation of motion is then

$$m\frac{d^2x}{dt^2} = -kx - R\frac{dx}{dt} \qquad (3)$$

This equation was solved in the previous lab and we note that when released the weight will oscillate with decreasing amplitude until it comes to rest at $x = 0$. If the damping is too large, it won't even oscillate but instead will slowly approach its equilibrium position.

For this experiment we are interested in a more complicated case where the top of the spring is moved up and down at an angular frequency ω by an external agent. This periodic displacement of the top of the spring translates into a periodic force on the weight

$$F = F_0\sin\omega t \qquad (4)$$

and the equation of motion becomes

$$m\frac{d^2x}{dt^2} = F_0\sin\omega t - kx - R\frac{dx}{dt} \qquad (5)$$

When the force is initially switched on, the system undergoes some transient motion which is eventually damped out and the system settles into a steady state motion given by

$$x = A\sin(\omega t + \phi) \qquad (6)$$

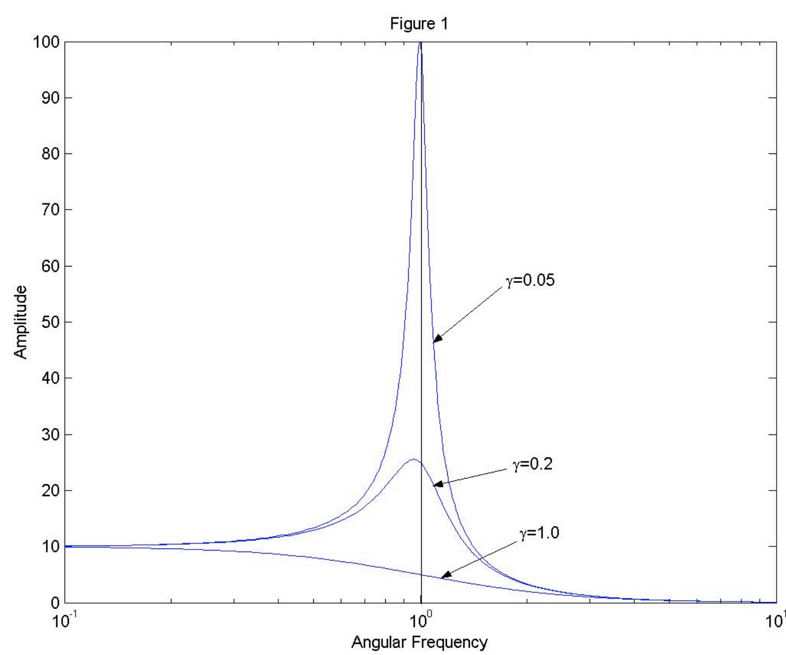
where the amplitude A is given by

$$A = \frac{F_0}{m} \Big/ \sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \qquad (7)$$

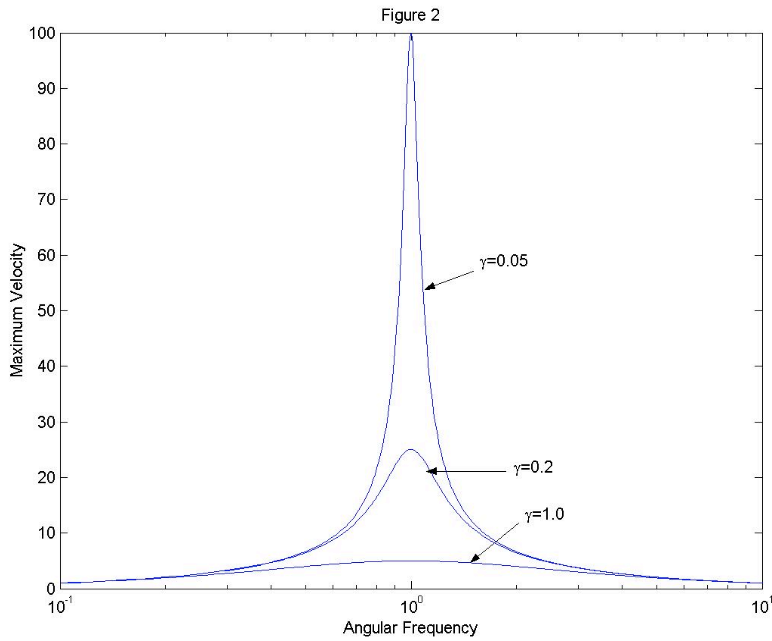
and the phase shift is

$$\tan\phi = 2\gamma\omega \big/ (\omega^2 - \omega_0^2) \qquad (8)$$

with $\omega_0 = \sqrt{k/m}$ and $\gamma = R/2m$. γ is called the damping constant. [Check that Eq.(6) is a solution to Eq.(5) by substituting it into Eq.(5).] Equation (7) is plotted in Fig. 1 (for $\omega_0 = 1$). A is resonant, that is it has a maximum, at $\omega = \omega_0$, because the denominator in Eq.(7)reaches a minimum (actually the minimum occurs when $\omega = \sqrt{\omega_0^2 - 2\gamma^2} \equiv \omega_R$, but for this experiment $\gamma << \omega_0$ and we will not be able to detect this shift). Note that if there were no damping ($\gamma = 0$), A would be infinite at $\omega = \omega_0$. In practice there always is some damping and the amplitude remains finite.

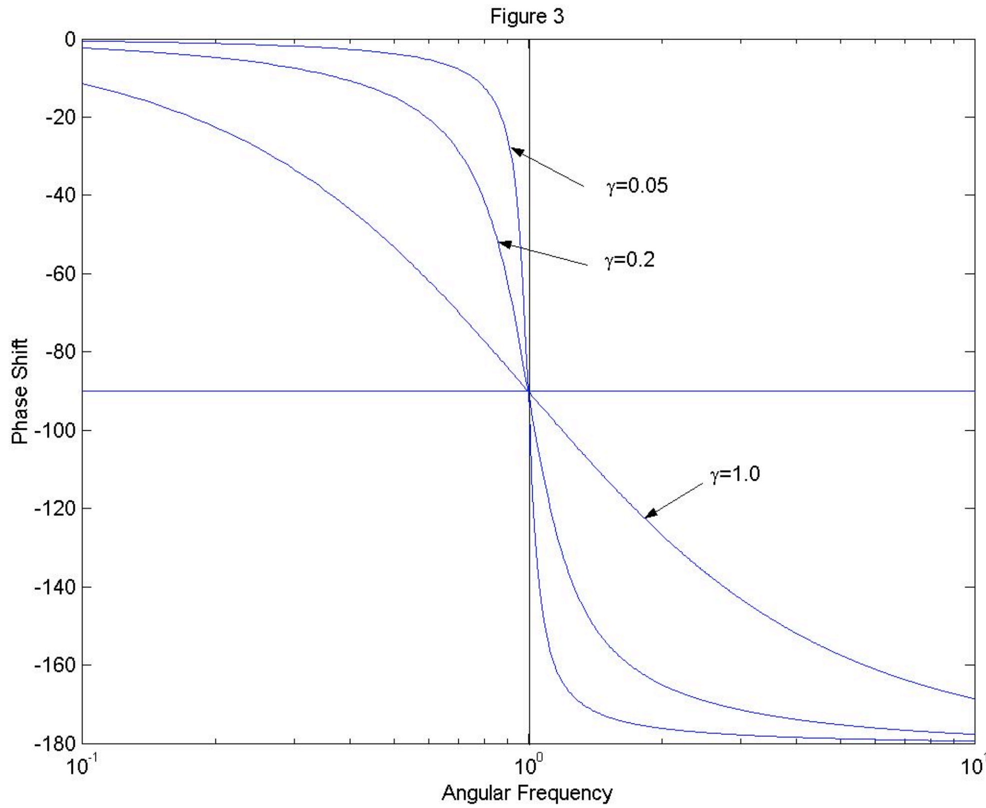


Another quantity of interest is $v_{max} = \omega A$ which is obtained directly from Eq.(6) since $v = dx/dt$. As shown in Fig. 2, v_{max} is symmetric about ω_0 if it is plotted on a log scale (i.e. v_{max} vs. $\log\omega$).



The phase shift ϕ (see Fig. 3) gives the phase difference between the driving force and the displacement of the mass. This force reaches a maximum at a time to given by $\omega t_0 = \pi/2$ (i.e. $\sin\omega t_0 = 1$), while x reaches a maximum at a time t_1 given by $\omega t_1 + \phi = \pi/2$. So $t_1 = t_0 - \phi/\omega$. Because ϕ is negative as shown in Fig. 3, $t_1 > t_0$ and the displacement lags the force (i.e. reaches a maximum after the force does), we see that for very low frequencies x and F are almost in phase. But for higher frequencies x lags until it is exactly out of phase (180º) at very high frequencies.

We can likewise derive the phase shifts between the force and velocity and find that v lags F by $\pi/2$ for low ω, is exactly in phase with F at $\omega = \omega_0$, then leads by $\pi/2$ for large ω.



These phase relations-are important for determining how much work F does on the mass. Since we are considering the steady state solution, the energy in the system does not change with time. So the work done by the external force in one cycle equals the energy dissipated by the damping force. This is easily calculated since the power P supplied by F is

$$P = \frac{\Delta W}{\Delta t} = \frac{F(t)\Delta x}{\Delta t} = F(t)v(t)$$

The average over one cycle is

$$< P >= \frac{1}{\tau}\int_0^\tau P(t)dt = \frac{1}{\tau}\int_0^\tau F(t)v(t)dt$$

where $\tau = 2\pi/\omega$ is the period. Using Eqs. (4) and (6) we find

$$< P >= \frac{A\omega}{2} F_0 \cos\theta \qquad (9)$$

where $\theta = \phi - \pi/2$ is the phase difference between F and v. $\cos\theta$ is called the power factor. Using Eqs.(7) and (8) we can show

$$< P >= \gamma m (A\omega)^2 = \gamma m v_{max}^2 \tag{10}$$

We see from Fig.(2) that at resonance the power supplied is a maximum and has the value (for $\gamma << \omega_0$)

$$< P >_{max} = \frac{F_0^2}{4\gamma m} \tag{11}$$

The Quality Factor, or Q, is a very important parameter used to describe a resonance. It is defined to be

$$Q = \frac{\omega_R}{2\gamma} \tag{12}$$

and is also (for $\gamma << \omega_0$)

$$Q = 2\pi \frac{\text{(maximum energy stored at resonance)}}{\text{(energy dissipated at resonance in one cycle)}}$$

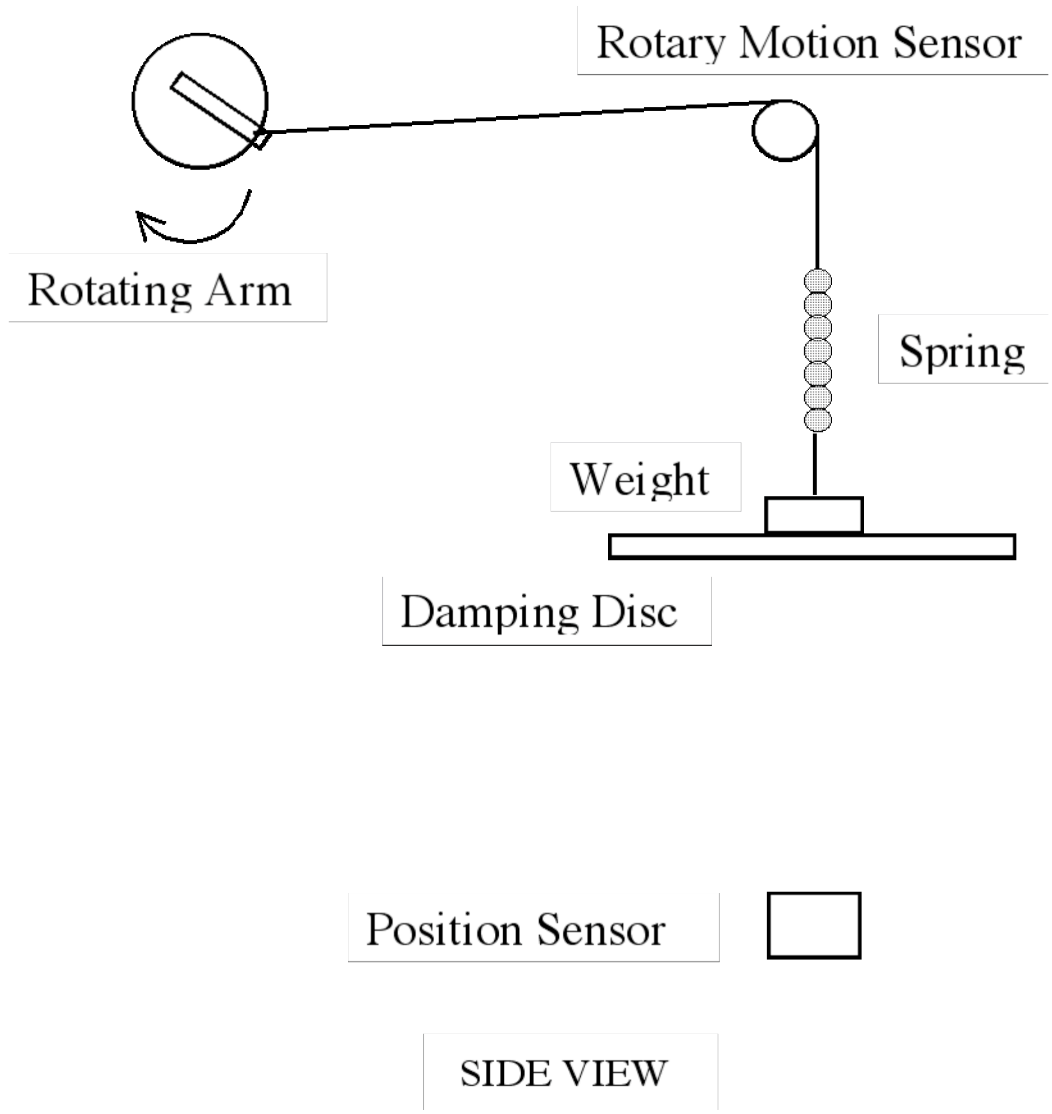
An alternative (equivalent) definition is that is more useful experimentally is (for $\gamma << \omega_0$)

$$Q = \frac{\omega_0}{2\Delta\omega} \tag{13}$$

where $2\Delta\omega = \omega_+ - \omega_-$ and $\omega_\pm$ are the frequencies on either side of ω_0 where $< P >= \frac{1}{2} < P >_{max}$. $2\Delta\omega$ is called the full width at half power of the resonance. Experimentally one usually does not measure $< P >$ to get $2\Delta\omega$. Instead one can use the data for v_{max} since by Eq.(10), $v = \frac{1}{\sqrt{2}} v_{max} = 0.707 v_{max}$. So $2\Delta\omega$ is obtained from Fig. 2 by determining the two frequencies where the velocity drops to 71% of its maximum value.

APPARATUS

The apparatus you will use is shown schematically below:



The rotating arm is driven by an electric motor whose speed is controlled by the voltage supplied by a variable power supply. Record the voltage applied to the motor at each speed setting so that a repeat of a setting is possible.

The rotary motion sensor shows the movement of the top of the spring, so its oscillations have the same phase as the external force applied to the system of the mass on spring.

The position sensor gives the position of the mass on the spring. Both angle sensor and position sensor should be read by Logger Pro as in the previous lab. From the Logger Pro displays, find the period of the oscillation and the amplitude A and phase ϕ of the oscillation of the mass; measure the phase as the difference between the position and the rotary motion sensor oscillations.

PROCEDURE

A. Measure the natural period of the system using a stop watch (on computer) or using Logger Pro. Time over 10 periods to reduce starting and stopping errors and repeat the measurement several times so that you have an estimate of your error. The available drive angular frequency ranges from about 10 rad/s to about 3 rad/s. If necessary, adjust the mass to set the natural angular frequency to about 5-6 rad/s.

B. Measure the amplitude and phase of the oscillation at each motor frequency as that frequency is varied over its entire range. For each frequency setting measure the period of the motor. Make your measurements very carefully, taking closely spaced frequency points near the peak of the resonance where the amplitude and phase are changing rapidly. For each measurement allow time for the system to reach steady state after you changed the rotation speed.

C. Use MatLab to make plots of A vs. ω, $v_{\max}$ vs. ω, and ϕ vs. ω. From your graph of $v_{\max}$ determine ω_0 and estimate your error. [$v_{\max}$ is calculated from the experimental data using the equation $v_{\max} = \omega A$.] Compare this value of ω_0 with the natural frequency you determined in part A. Also from your plot of $v_{\max}$, estimate $2\Delta\omega$ from the points where $v = 0.7 \times \text{peak}(v_{\max})$. Estimate the error. Use Eq.(13) to determine Q and Eq.(12) to determine γ and R. Estimate your errors.

D. Using the values ω_0 and γ you determined in part D, use MatLab to calculate the theoretical frequency dependence of $v_{\max}$ and ϕ. Add these curves to the plots you prepared in part D.

E. In part D you used a rather crude procedure to determine the parameters γ and ω_0. What, in general terms, would be a better way to fit the theory to the experiment?

F. Your lab report should include a description of the apparatus and measurements. Give a clear discussion of each of your results from sections B-E. Answer all questions and give all information requested.

HINTS FOR USING LOGGER PRO

To setup the program, go to:

Menu EXPERIMENT

- CONNECT INTERFACE
- CONNECT ON PORT: select COM1

Button LABPRO

- Drag MOTION DETECTOR to DIG/SONIC2 box
- Drag Rotary Motion to DIG/SONIC1 box
- CLOSE

Menu EXPERIMENT

- DATA COLLECTION …
- Tab COLLECTION
- LENGTH 10 seconds
- Sample at Time Zero: off
- SAMPLING RATE about 20 samples per sec
- DONE

MATLAB NOTE:

The atan() function returns values in a range of angles that is wrong for our definition of the phase shift (Eqn 8). To make the angles in array phi = atan(…) fall in the correct range, use:

```
k = find( phi > 0);

phi(k) = phi(k) – pi;
```

Explain in your report the difference between ranges of phi.