0.9 show three and two points, respectively, because of the more complex motion. The number of points n on the Poincaré section here shows that the new period $T = T_0 n/m$, where $T_0 = 2\pi/\omega$ is the period of the driven force and m is an integer ($m = 2$ for the $F = 0.5$ plot and $m = 1$ for the $F = 0.9$ plot). The chaotic motions for $F = 0.6$, 0.7, and 1.0 display the complicated variation of points expected for chaotic motion with a period $T \to \infty$. The Poincaré sections are also rich in structure for chaotic motion.

On three occasions thus far (Figures 4-5, 4-7, and 4-11), we have pointed out *attractors*, a set of points (or a point) on which the motion converges for dissipative systems. The regions traversed in phase space are strictly bounded when there is an attractor. In chaotic motion, nearby trajectories in phase space are continually diverging from one another but must eventually return to the attractor. Because the attractors in these chaotic motions, called *strange* or *chaotic attractors*, are necessarily bounded in phase space, the attractors must fold back into the nearby regions of phase space. Strange attractors create intricate patterns, because the folding and stretching of the trajectories must occur such that no trajectory in phase space intersects, which is ruled out by the deterministic dyamical motion. The Poincaré sections of Figure 4-19 reveal the folded, layered structure of the attractors. Chaotic attractors are fractals, but space does not permit further discussion of this extremely interesting phenomenon.

4.7 Mapping

Logistic equation.

If we use n to denote the time sequence of a system and x to denote a physical observable of the system, we can describe the progression of a nonlinear system at a particular moment by investigating how the $(n + 1)$th state (or iterate) depends on the nth state. An example of such a simple, nonlinear behavior is $x_{n+1} = (2x_n + 3)^2$. This relationship, $x_{n+1} = f(x_n)$, is called **mapping** and is often used to describe the progression of the system. The Poincaré section plots previously discussed are examples of two-dimensional maps. A physical example appropriate for mapping might be the temperature of the space shuttle orbiter tiles while the shuttle descends through the atmosphere. After the orbiter has been on the ground for some time, the temperature T_{n+1} is the same as T_n, but this was not true while the shuttle plummeted through the atmosphere from its earth orbit. Modeling the tile temperatures correctly with a mathematical model is difficult, and linear assumptions are often first assumed in such calculations with nonlinear terms added to make more realistic calculations.

We can write a *difference equation* using $f(\alpha, x_n)$ where x_n is restricted to a real number in the interval (0, 1) between 0 and 1, and α is a model-dependent parameter.

$$x_{n+1} = f(\alpha, x_n) \tag{4.44}$$

The function $f(\alpha, x_n)$ generates the value of x_{n+1} from x_n, and the collection of points generated is said to be a **map** of the function itself. The equations, which are often nonlinear, are amenable to numerical solution by iteration, starting with x_1. We will restrict ourselves here to one-dimensional maps, but two-dimensional (and higher order) equations are possible.

Mapping can best be understood by looking at an example. Let us consider the "logistic" equation, a simple one-dimensional equation given by

$$f(\alpha, x) = \alpha x(1 - x) \tag{4.45}$$

so that the iterative equation becomes

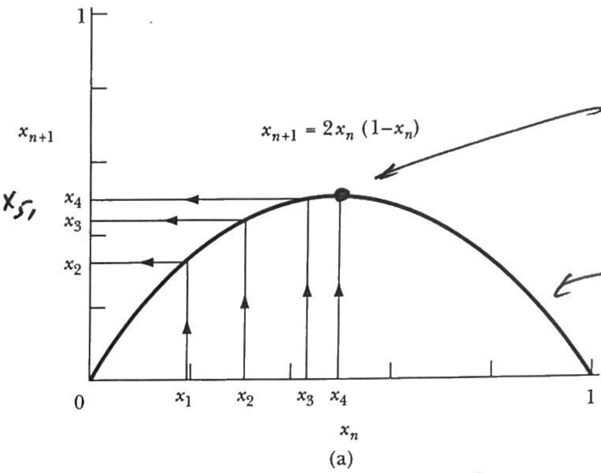
$$x_{n+1} = \alpha x_n(1 - x_n) \tag{4.46}$$

We follow the discussion of Bessoir and Wolf (Be91) who use the logistic equation for a biological application example of studying the population growth of fish in a pond, where the pond is well isolated from external effects such as weather. The iterations, or n values, represent the annual fish population, where x_1 is the number of fish in the pond at the beginning of the first year of the study. If x_1 is small, the fish population may grow rapidly in the early years because of available resources, but overpopulation may eventually deplete the number of fish. The population x_n is scaled so that its value fits in the interval (0, 1) between 0 and 1. The factor α is a model-dependent parameter representing average effects of environmental factors (e.g., fishermen, floods, drought, predators) that may affect the fish. The factor α may be varied as desired in the study, but experience shows that α should be limited in this example to the interval (0, 4) to prevent the fish population from becoming negative or infinite.

The results of the logistic equation are most easily observed by graphical means in a map called the *logistic map*. The iteration x_{n+1} is plotted versus x_n in Figure 4-21a for a value of $\alpha = 2.0$. Starting with an initial value x_1 on the horizontal (x_n) axis, we move up until we intersect with the curve $x_{n+1} = 2x_n(1 - x_n)$, and then we move to the left where we find x_2 on the vertical axis (x_{n+1}). We then start with this value of x_2 on the horizontal axis and repeat the process to find x_3 on the vertical axis. If we do this for a few iterations, we converge on the value $x = 0.5$, and the fish population stabilizes at half its maximum. We arrive at this result independent of our initial value of x_1 as long as it is not 0 or 1.

An easier way to follow the process is to add the 45° line, $x_{n+1} = x_n$, to the same graph. Then after initially intersecting the curve from x_1, one moves horizontally to intersect with the 45° line to find x_2 and then moves up vertically to find the next iterative value of x_3. This process can go on and reach the same result as in Figure 4-21a. We show the process in Figure 4-21b to indicate that this method is easier to use than the one without the 45° line.

In practice, we want to study the behavior of the system when the model parameter α is varied. In the present case, for values of α less than 3.0, stable populations will result (Figure 4-22a). The solutions follow a square spiral path to the central, final value. For values of α just above 3.0, more than one solution for the fish population occurs (Figure 4-22b). The solutions follow a path similar to the square spiral, which converges to the two points at which the square intersects the "iteration line," rather than to a single point. Such a change in the number of solutions to an equation, when a parameter such as α is varied, is called **a bifurcation**.



. . . . . x_6, x_5,

converges here.

This line is just a plot $y(x) = 2x(1-x)$ for $x = 0 \div 1$.

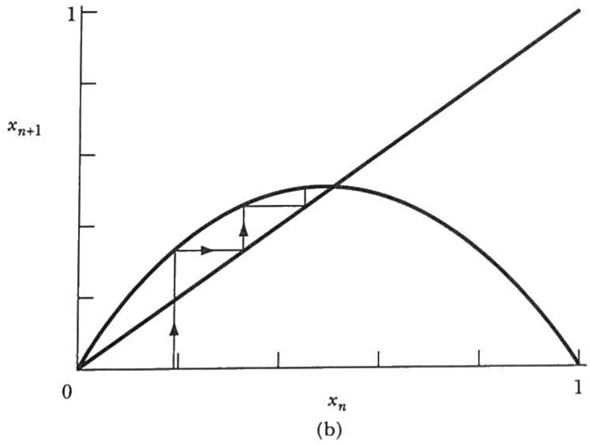


FIGURE 4-21 Techniques for producing a map of the logistics equation.

We obtain a more general view of the global picture by plotting a *bifurcation diagram,* which consists of x_n, determined after many iterations to avoid initial effects, plotted as a function of the model parameter α. Many new interesting effects emerge indicating regions and *windows* of stability as well as those of chaotic dynamics. We show the bifurcation diagram in Figure 4-23 for the logistic equation over the range of α values from 2.8 to 4.0. For the value of $\alpha = 2.9$ shown in Figure 4-22a, we observe that after a few iterations, a stable configuration for $x = 0.655$ results. An *N cycle* is an orbit that returns to its original position after N iterations, that is, $x_{N+i} = x_i$. The period for $\alpha = 2.9$ is then a *one cycle.* For $\alpha = 3.1$ (Figure 4-22b), the value of x oscillates between 0.558 and

$x_{n\to\infty}(\alpha)$.

bifurcation diagram.

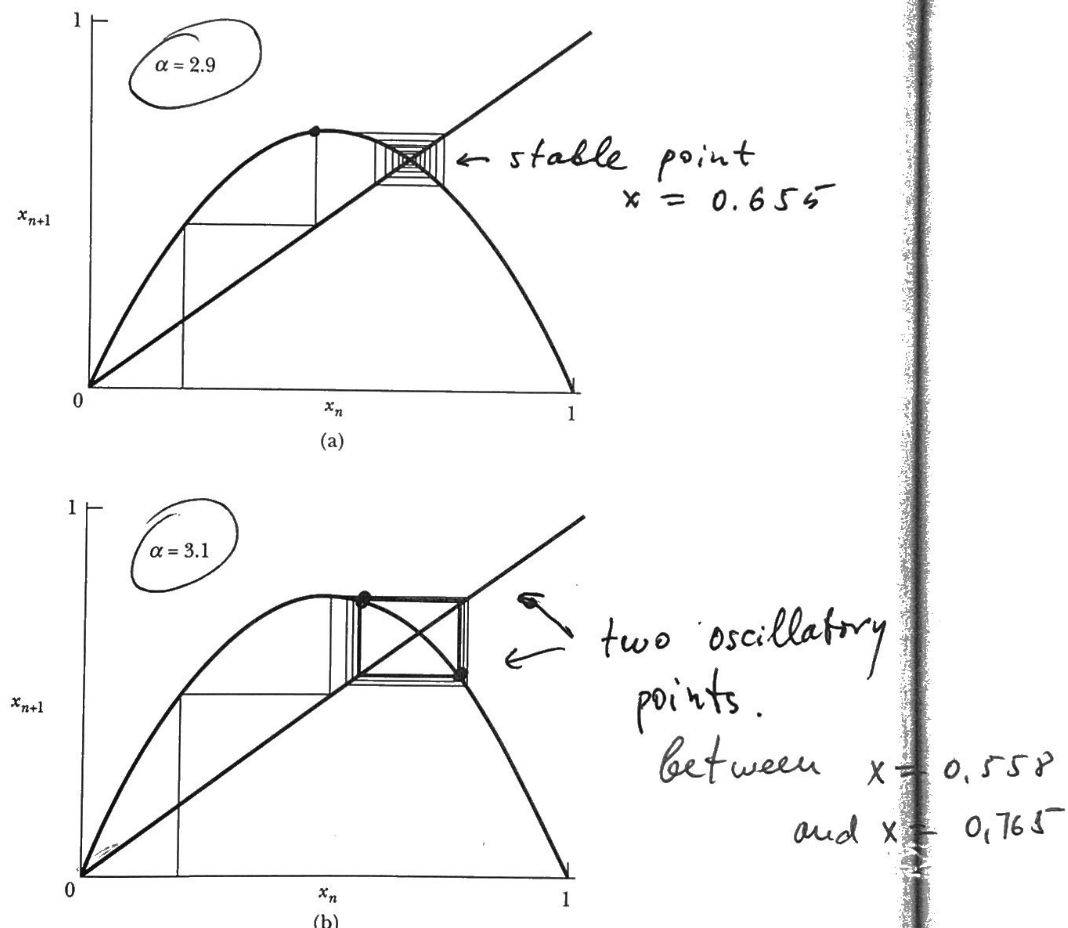


FIGURE 4-22 Logistic equation map for α values of 2.9 and 3.1, indicating stable populations in (a) and multiple possible solutions for $\alpha > 3.0$ in (b).

0.765 (two cycle) after a few iterations evolve. The bifurcation occuring at 3.0 is called a *pitchfork bifurcation* because of the obvious shape of the diagram caused by the splitting. At $\alpha = 3.1$, the period doubling effect has $x_{n+2} = x_n$. At $\alpha = 3.45$, the two-cycle bifurcation evolves into a four cycle, and the bifurcation and period doubling continues up to an infinite number of cycles near $\alpha = 3.57$. Chaos occurs for many of the α values between 3.57 and 4.0, but there are still windows of periodic motion, with an especially wide window around 3.84. A really interesting behavior occurs for $\alpha = 3.82831$ (Problem 4-11). An apparent periodic cycle of 3 years seems to occur for several periods, but then it suddenly violently changes for a few years, and then returns again to the 3-year cycle. This *intermittent* behavior could certainly prove devastating to a biological study operating over several years that suddenly turns chaotic without apparent reason.

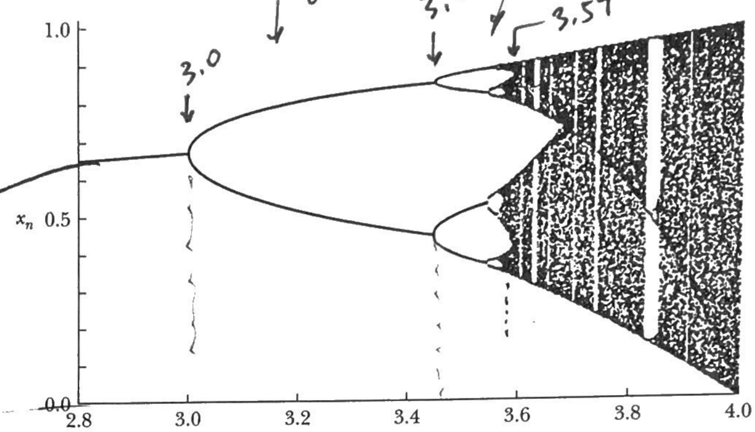


FIGURE 4-23 Bifurcation diagram for the logistic equation map.

a

EXAMPLE 4.2

Let $\Delta\alpha_n = \alpha_n - \alpha_{n-1}$ be the width between successive period doubling bifurcations of the logistic map that we have been discussing. For example, from Figure 4-23, we let $\alpha_1 = 3.0$ where the first bifurcation occurs and $\alpha_2 =$ 3.449490 where the next one occurs. Let δ_n be defined as the ratio

$$\delta_n = \frac{\Delta\alpha_n}{\Delta\alpha_{n+1}} \tag{4.47}$$

and let $\delta_n \rightarrow \delta$ as $n \rightarrow \infty$. Find δ_n for the first few bifurcations and the limit δ.

Solution. Although we could program this numerical calculation with a computer, we will use one of the commercially available software programs (Be91) to work this example. We make a table of the α_n values using the computer program, find $\Delta\alpha_n$, and then determine a few values of α_n.

n	α_n	$\Delta\alpha$	δ_n
1	3.0		
2	3.449490	0.449490	4.7515
3	3.544090	0.094600	4.6562
4	3.564407	0.020317	4.6684
5	3.568759	0.004352	
∞	3.5699456		4.6692

As α_n approaches the limit 3.5699456, the number of period doublings approaches infinity, and the ratio δ_n, called *Feigenbaum's number,* approaches 4.669202. This result was first found by Mitchell Feigenbaum in the 1970s, and he found that the limit δ was a universal property of the period doubling route to chaos when the function $f(\alpha, x)$ has a quadratic maximum. It is a remarkable fact that this universality is not confined to one-dimensional mappings; it is also true for two-dimensional maps and has been confirmed for several cases. Feigenbaum claims to have found this result using a programmable hand calculator. The calculation obviously has to be carried to many significant figures to establish its accuracy, and such a calculation was not possible before such calculators (or computers) were available.

4.8 Chaos Identification

In our driven and damped pendulum, we found that chaotic motion occurs for some values of the parameters, but not for others. What are the characteristics of chaos and how can we identify them? Chaos does not represent periodic motion, and its limiting motion will not be periodic. Chaos can generally be described as having a sensitive dependence on initial conditions. We can demonstrate this effect by the following example.

EXAMPLE 4.3

Consider the nonlinear relation $x_{n+1} = f(\alpha, x_n) = \alpha x_n(1 - x_n^2)$. Let $\alpha = 2.5$ and make two numerical calculations with initial x_1 values of 0.700000000 and 0.700000001. Plot the results and find the iteration n where the solutions have clearly diverged.

Solution. The iterative equation that we are considering is

$$x_{n+1} = \alpha x_n(1 - x_n^2) \tag{4.48}$$

We perform a short numeric calculation and plot the results of iterations for the two initial values on the same graph. The result is shown in Figure 4-24 where there is no observed difference for x_{n+1} until n reaches at least 30. By $n = 39$, the difference in the two results is marked, despite the original values differing by only 1 part in 10^8.

If the computations are made without error, and the difference between iterated values doubled on the average for each iteration, then there will be an exponential increase such as

$$2^n = e^{n \ln 2}$$

where n is the number of iterations undergone. For the iterates to be separated by the order of unity (the size of the attractor), we will have

$$2^n 10^{-8} \sim 1$$