The birth of topological insulators

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Certain insulators have exotic metallic states on their surfaces. These states are formed by topological effects that also render the electrons travelling on such surfaces insensitive to scattering by impurities. Such topological insulators may provide new routes to generating novel phases and particles, possibly finding uses in technological applications in spintronics and quantum computing.

Many aspects of condensed-matter physics are concerned with understanding how order emerges when a very large number of simple constituents, such as ions, magnetic moments or electrons, interact with each other. In ordered phases such as crystals and magnets, the order is described through symmetry breaking: in a crystal, ions are arranged periodically owing to their electrostatic interactions, thereby breaking the continuous symmetry of space under rotations and translations; in typical magnets, some of the rotational symmetry of spin space is broken, together with the time-reversal symmetry.

A major discovery in the 1980s was that electrons that are confined to two dimensions and subject to a strong magnetic field show a completely different, topological, type of order, which underlies the quantum Hall effect. Consequences of this order include dissipationless transport and emergent particles with fractional charge and statistics. One of the important discoveries of the past few years is that topological order also occurs in some three-dimensional (3D) materials; in these materials, the role of the magnetic field is assumed by the mechanism of spin–orbit coupling, an intrinsic property of all solids. These materials have been named topological insulators because they are insulators in the 'bulk' but have exotic metallic states present at their surfaces owing to the topological order.

In this Perspective article, I provide an overview of the basic concepts underlying topological insulators and recent studies of these remarkable new materials. After an explanation of what makes some insulators 'topological' and a brief history of this rapidly developing field, I discuss recent advances made in experiments on topological insulators — for both bulk and nanostructured materials — and in the theoretical understanding of these materials. I conclude by explaining why many research groups are seeking to use topological insulators to generate new particles and phases, which has possible applications to quantum computing.

A primer on topological insulators

The easiest way to describe a topological insulator is as an insulator that always has a metallic boundary when placed next to a vacuum or an 'ordinary' insulator. These metallic boundaries originate from topological invariants, which cannot change as long as a material remains insulating. An intuitive illustration showing why the metallic surfaces exist is presented in Fig. 1a: a trefoil knot is used to represent a topological insulator and a closed loop to represent an ordinary one. Topology is a branch of mathematics that studies the properties of objects that are invariant under smooth deformations, a classic example being a doughnut transforming into a coffee cup. In contrast to the doughnut/coffee cup pair, the trefoil knot and the closed loop have different topological invariants and thus neither can be deformed to become the other, no matter how the string (or wire) is stretched or

twisted, without being cut. Nevertheless, these invariants do change in crossing the interface between topological and ordinary insulators, so by contradiction the surface cannot remain insulating, which is analogous to cutting the knot.

In a topological insulator, the component that is 'knotted' is the electron's wavefunction as it moves through momentum space. Associated with this knotting are topological invariants (usually expressed as integrals involving the wavefunction) that cannot change as long as the material remains insulating. Even when the boundary between ordinary insulators and topological insulators is atomically sharp and the continuous description using topology might not seem to be applicable, there are nevertheless metallic delocalized wavefunctions at this boundary. In Fig. 1b, the simplest case of knotting in a 3D electronic structure is illustrated: for one occupied electronic band and one empty one, each point in 3D momentum space is associated with a unit vector representing the occupied state, and the Hopf map shown in Fig. 1b is a topologically non-trivial example.

Lessons from the past

The appearance of the unusual metal when the topology changes at a surface is the main experimental signature that an insulator is indeed topological. To explain the properties of the insulator, I will begin with a short review of the historical developments that led to the theoretical predictions that topological insulators exist. A simpler version of this metal occurs at the edge of a quantum Hall droplet, the first reported example of two-dimensional (2D) topological order. Quantum Hall edges are perfect quantum wires wrapping around the insulating droplet (Fig. 2a), and they result from the topological properties of the electronic wavefunctions when the constituent electrons are confined to two dimensions and subject to a strong magnetic field perpendicular to the plane to which the electrons are confined (see page 187 for more details on the quantum Hall effect).

Work on topological insulators grew out of the idea that the quantum Hall effect that arises in such 2D systems in the presence of a magnetic field could occur even for electrons moving on a lattice in the absence of a macroscopic magnetic field[1]. Instead of being driven by such a magnetic field, it was predicted in the late 1980s that electrons could, in principle, form a quantum Hall state driven by forces that result from their motion through the crystal lattice. Recent developments are based on spin–orbit coupling, a relativistic effect in which the spin and orbital angular momentum degrees of freedom of electrons are coupled; this coupling causes electrons that are moving through a crystal to feel a spin-dependent force, even in non-magnetic materials.

Although spin–orbit coupling does not have the symmetry required to induce the quantum Hall effect (that is, it does not break time-reversal

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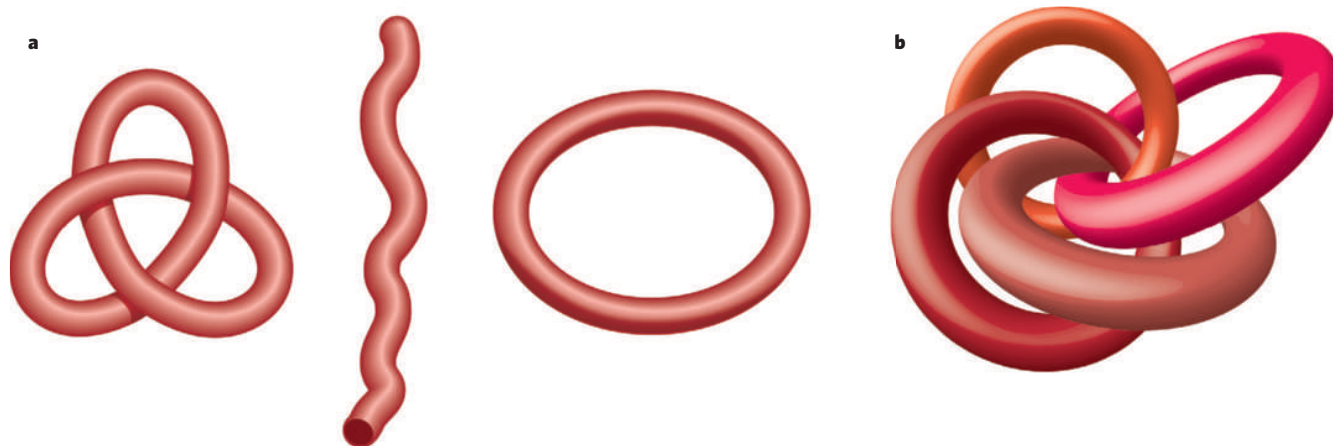
a   **b**

Figure 1 | **Metallic states are born when a surface unties 'knotted' electron wavefunctions. a**, An illustration of topological change and the resultant surface state. The trefoil knot (left) and the simple loop (right) represent different insulating materials: the knot is a topological insulator, and the loop is an ordinary insulator. Because there is no continuous deformation by which one can be converted into the other, there must be a surface where the string is cut, shown as a string with open ends (centre), to pass between the two knots; more formally, the topological invariants cannot remain defined. If the topological invariants are always defined for an insulator, then the surface must be metallic. **b**, The simplest example of a knotted 3D electronic band structure (with two bands)[35], known to mathematicians as the Hopf map. The full topological structure would also have linked fibres on each ring, in addition to the linking of rings shown here. The knotting in real topological insulators is more complex as these require a minimum of four electronic bands, but the surface structure that appears is relatively simple (Fig. 3).

symmetry as an applied magnetic field would), in simplified models introduced in around 2003 it can lead to a quantum spin Hall effect, in which electrons with opposite spin angular momentum (commonly called spin up and spin down) move in opposite directions around the edge of the droplet in the absence of an external magnetic field[2] (Fig. 2b). These simplified models were the first steps towards understanding topological insulators. But it was unclear how realistic the models were: in real materials, there is mixing of spin-up and spin-down electrons, and there is no conserved spin current. It was also unclear whether the edge state of the droplet in Fig. 2b would survive the addition of even a few impurities.

In 2005, a key theoretical advance was made by Kane and Mele[3]. They used more realistic models, without a conserved spin current, and showed how some of the physics of the quantum spin Hall effect can survive. They found a new type of topological invariant that could be computed for any 2D material and would allow the prediction of whether the material had a stable edge state. This allowed them to show that, despite the edge not being stable in many previous models, there are realistic 2D materials that would have a stable edge state in the absence of a magnetic field; the resultant 2D state was the first topological insulator to be understood. This non-magnetic insulator has edges that act like perfectly conducting one-dimensional electronic wires at low temperatures, similar to those in the quantum Hall effect.

Subsequently, Bernevig, Hughes and Zhang made a theoretical prediction that a 2D topological insulator with quantized charge conductance along the edges would be realized in (Hg,Cd)Te quantum wells[4]. The quantized charge conductance was indeed observed in this system, as a quantum-Hall-like plateau in zero magnetic field, in 2007 (ref. 5). These experiments are similar to those on the quantum Hall effect in that they require, at least so far, low temperature and artificial 2D materials (quantum wells), but they differ in that no magnetic field is needed.

Going 3D
The next important theoretical development, in 2006, was the realization[6–8] that even though the quantum Hall effect does not generalize to a genuinely 3D state, the topological insulator does, in a subtle way. Although a 3D 'weak' topological insulator can be formed by layering 2D versions, similar to layered quantum Hall states, the resultant state is not stable to disorder, and its physics is generally similar to that of the 2D state. In weak topological insulators, a dislocation (a line-like defect in the crystal) will always contain a quantum wire like that at the edge of the quantum spin Hall effect (discussed earlier), which may allow 2D topological insulator physics to be observed in a 3D material[9].

There is also, however, a 'strong' topological insulator, which has a more subtle relationship to the 2D case; the relationship is that in two dimensions it is possible to connect ordinary insulators and topological insulators smoothly by breaking time-reversal symmetry[7]. Such a continuous interpolation can be used to build a 3D band structure that respects time-reversal symmetry, is not layered and is topologically nontrivial. It is this strong topological insulator that has protected metallic surfaces and has been the focus of experimental activity.

Spin–orbit coupling is again required and must mix all components of the spin. In other words, there is no way to obtain the 3D strong topological insulator from separate spin-up and spin-down electrons, unlike in the 2D case. Although this makes it difficult to picture the bulk physics of the 3D topological insulator (only the strong topological insulator will be discussed from this point), it is simple to picture its metallic surface[6].

The unusual planar metal that forms at the surface of topological insulators 'inherits' topological properties from the bulk insulator. The simplest manifestation of this bulk–surface connection occurs at a smooth surface, where momentum along the surface remains well defined: each momentum along the surface has only a single spin state at the Fermi level, and the spin direction rotates as the momentum moves around the Fermi surface (Fig. 3). When disorder or impurities are added at the surface, there will be scattering between these surface states but, crucially, the topological properties of the bulk insulator do not allow the metallic surface state to vanish — it cannot become localized or gapped. These two theoretical predictions, about the electronic structure of the surface state and the robustness to disorder of its metallic behaviour, have led to a flood of experimental work on 3D topological insulators in the past two years.

Experimental realizations
The first topological insulator to be discovered was the alloy Bi_xSb_{1-x}, the unusual surface bands of which were mapped in an angle-resolved photoemission spectroscopy (ARPES) experiment[10,11]. In ARPES experiments, a high-energy photon is used to eject an electron from a crystal, and then the surface or bulk electronic structure is determined from an analysis of the momentum of the emitted electron. Although the surface structure of this alloy was found to be complex, this work launched a search for other topological insulators.
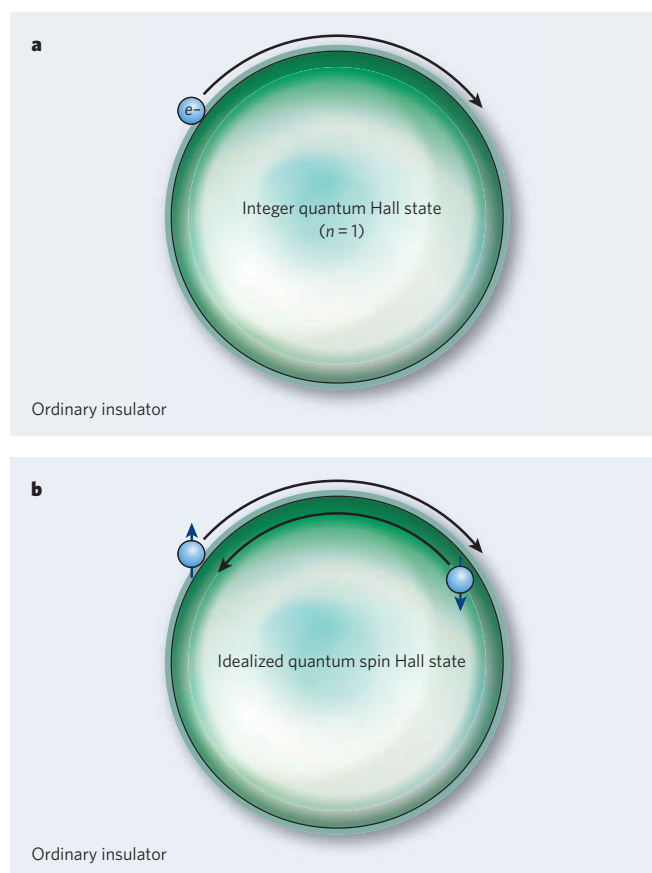
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Figure 2 | Topological order in two dimensions. a, Edge of an integer quantum Hall state. The electrons (e^-) are confined to a 2D insulating droplet with a metallic edge. Along the edge, electrons propagate only in one direction, which is determined by the sign of the applied magnetic field perpendicular to the droplet. One integer, n, the topological invariant, determines the Hall conductance and the number of propagating edge modes. **b,** Edge of an idealized quantum spin Hall state (that is, a 2D topological insulator). Along the edge, spin-up electrons move clockwise, whereas spin-down electrons move anticlockwise. Spin-up and spin-down electrons are independent and are in oppositely directed quantum Hall states. An applied electrical field generates a spin current but no charge current. Each droplet is surrounded by an ordinary insulator.

For a topological insulator to form, spin–orbit coupling must be strong enough to modify the electronic structure significantly, which suggests that heavy-element, small-bandgap semiconductors are the most promising candidates. This suggestion stems from two points. First, spin–orbit coupling is a relativistic effect and is only strong for heavy elements. Second, if the bandgap is much larger than the energy scale of spin–orbit coupling, then spin–orbit coupling will not be able to change the phase. The search for topological insulators culminated in the recent discovery of topological insulator behaviour in Bi_2Se_3 and Bi_2Te_3 (refs 12–14). These 'next-generation' materials both show topological insulator behaviour up to higher temperatures than does the original material (Bi_xSb_{1-x}), with bulk bandgaps of more than 0.1 eV, and have the simplest surface state that is allowed. Beyond providing further confirmation of the theory of topological insulators, the simplicity of the surface state in these materials opens up the possibility of many experiments, some of which are described below. Furthermore, the large bandgap means that these experiments do not need to be carried out at extremely low temperatures. The main remaining complication about these materials, especially when using experimental techniques that do not distinguish directly between bulk states and surface states (unlike ARPES), is that in the bulk state they have residual conductivity arising from impurities.

A graphene lookalike

Figure 3 shows the measured surface state of Bi_2Se_3 and a theoretical idealization of its state, including the electron spin. The surface state of the next-generation topological insulators is closely related to the Dirac electronic structure of graphene, which has a linear energy–momentum relationship like that of a relativistic particle (and is known as a Dirac cone). Graphene, which consists of a single layer of carbon atoms, has been an extremely active subject of research in recent years[15]: it is interesting both structurally, because it is the most 2D material possible, and electronically, because of its linear energy–momentum relationship. The main difference between the surface of a topological insulator and that of graphene is that the topological insulator has only one Dirac point (or valley) and no spin degeneracy, whereas graphene has two Dirac points and is spin degenerate. This difference has far-reaching consequences, including the possibility of generating new particles that have applications in quantum computing.

Another remarkable consequence of the absence of the extra degeneracies was detected by scanning tunnelling microscopy experiments[16–18]: the interference patterns near defects or steps on the surface show that electrons are never completely reflected when scattered. Even if the disorder becomes much stronger and a description in terms of well-separated scattering events is invalid, the surface remains metallic[19]. This protection of the surface metal from Anderson localization (that is, formation of an insulating state as a result of strong disorder[20]) is one of the key differences between the surface of the topological insulator and the 'accidental' surface states present in other materials, such as the noble metals. Beyond just being stable to disorder, it is now understood that the topological insulator, at least in some 2D models, forms as a result of disorder[21,22]. In graphene, there is an approximate version of this protection if the scattering has a very smooth potential, but real graphene is likely to become localized with strong disorder.

There was, however, initially a disadvantage in using a topological insulator for some purposes as opposed to graphene. In graphene, the chemistry of the carbon atoms naturally locates the Fermi level at the Dirac point (that is, the point at which the two cones intersect), where the density of states vanishes. This means that the density of carriers in graphene is highly tunable using an applied electrical field and allows applications of graphene in both basic science and microelectronics. The surface Fermi level of a topological insulator does not have any particular reason to sit at the Dirac point; however, through a combination of surface and bulk chemical modification, tuning to the Dirac point in Bi_2Se_3 was recently demonstrated[23]. This control of chemical potential is important for applications, as well as for a proposal to create a topological exciton condensate by biasing a thin film of topological insulator[24]. This condensate is a superfluid state of electron–hole pairs that would have a bound electronic state around superfluid vortices.

Materials challenges

Two recent advances in the nanostructuring of topological insulators also deserve a mention. Nanoribbons of Bi_2Se_3 have been synthesized, allowing observation of the Aharonov–Bohm effect in the metallic surface state when a magnetic field is applied along the long direction of the nanoribbon[25], and molecular beam epitaxy has been used to grow thin films of Bi_2Se_3 with controlled thickness down to a single unit cell[26]. Such nanostructures are essential for many of the proposed applications of topological insulators to spintronics and other fields. For example, the spin structure in Fig. 3 means that a charge current along the surface of the topological insulator automatically yields a non-zero spin density; a heterostructure that combines a topological insulator with a ferromagnet could allow the ferromagnet to be switched by passing a current through the topological insulator's surface, and this would be a new type of spin torque device for magnetic memory applications[27]. The next important steps are to carry out direct transport and optical experiments on the metallic surface state to measure its conductivity and spin properties, and for these experiments improved materials with reduced residual bulk conductivity might be required.

A nascent field

Several experiments that are under way are based on a different picture of the essential properties of a topological insulator, a picture that is connected to particle physics research from the 1980s. It has been known for many years that the magnetic field and electric field are coupled in the interior of some insulators. Topological insulators contain a special type of such coupling that is quantized and was originally discussed by particle physicists who were analysing the coupling of an axion particle to ordinary electric and magnetic fields.

Particle physics and superfluidity

Condensed-matter physicists often seek to define a phase of matter in terms of its response to some influence. For example, a solid is a material with a non-zero stiffness in response to applied shear force, and a superconductor is defined in terms of the Meissner effect (that is, by its expulsion of applied magnetic fields). Such definitions are connected directly to experimentation and are independent of details on the microscopic level. It turns out that the defining response of a topological insulator was almost worked out in the 1980s in an effort to understand the properties of axion electrodynamics, which differs from ordinary electrodynamics by the addition to the Lagrangian of a term that is proportional to the scalar product of the electric field and the magnetic field, $\mathbf{E} \cdot \mathbf{B}$ (ref. 28). There are only two values of the coefficient of this term that are compatible with time-reversal symmetry, and these correspond to ordinary insulators and topological insulators. Hence, topological insulators are materials whose internal structure generates the non-zero value of the axion-like coupling, in the same way that insulators modify the dielectric constant, which is the coefficient of $\mathbf{E}^2$ in the Lagrangian[29].

There are many predicted consequences of this term[30,31], including monopole-like behaviour and surface states with continuously variable Hall conductivity. The most important for applications may be the simple magnetoelectric effect in which an applied electrical field generates a magnetic dipole and vice versa. The potential advantage of the magnetoelectric effect in topological insulators over that in multiferroic materials[32] is an increase not in strength but instead in speed and reproducibility without fatigue, because this effect results purely from orbital motion of the electrons. Whether a given material is a topological insulator can be theoretically defined simply by asking how much its bulk polarization changes in an applied magnetic field, and measurements of this magnetoelectric polarizability in other materials suggest that it

should be possible to observe the effect in topological insulators.

Another recent theoretical advance was understanding how phases similar to the 3D topological insulator might appear in systems with different symmetries than those of the non-magnetic solids discussed above[33–35]. Finding experimental examples of these other types of topological insulator and superconductor would be a major accomplishment. Work in this area also led to a reconsideration of the properties of superfluid ³He, in which neutral fermionic atoms form Cooper pairs: the quasiparticles in this superfluid have some remarkable properties that are closely connected to those of electrons in the topological insulator, and finding direct experimental signatures of these quasiparticles is an important challenge.

Emergent particles and quantum computing

The interface between a topological insulator and a superconductor may allow the creation of an 'emergent' particle that neither material supports by itself. Like any other metal, the metallic surface of a topological insulator becomes superconducting, by way of the proximity effect, when placed next to an ordinary superconductor. However, the superconducting surface of a topological insulator is novel: if a vortex line runs from the superconductor into the topological insulator, then a zero-energy Majorana fermion is trapped in the vicinity of the vortex core[36,37]. Unlike vortex core states in ordinary superconductors, the Majorana fermion has quantum numbers that differ from those of an ordinary electron: it is its own antiparticle, is electrically neutral and is, in most respects, 'half' of an ordinary electron[38]. There are several reasons for the excitement about this proposal and others of a similar spirit[39]. First, these proposals may allow a Majorana fermion to be observed directly, a long-sought goal in particle physics and condensed-matter physics. Second, Majorana fermions are one step towards a topological quantum computer, which would be exceptionally well protected from errors[40] because the quasiparticles obey a special kind of quantum statistics that is non-Abelian (for a more in-depth discussion of non-Abelian states of matter, see page 187). More complex materials might even allow Majorana fermions as point excitations in the bulk of the 3D material[41].

More generally, there are several aspects of the topological order in quantum Hall states that have not yet been realized in topological insulators, and the most important is the emergence of quasiparticles with modified charge and statistics. The traditional wellspring of topological order, the 2D electron gas in a strong magnetic field, has not dried up;
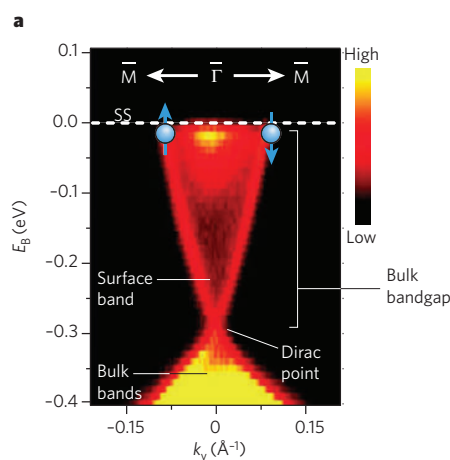

Figure 3 | Signatures of the exotic metallic surface states in topological insulators. a, The electronic structure of Bi_2Se_3, as measured by ARPES. Measured energy electron energy, E_B, is plotted against electron momentum, k_y. High intensity (red and yellow areas) indicates a non-zero electronic density of states. The surface bands crossing the bulk bandgap enclose a single Dirac point at the Brillouin-zone centre ($\bar{\Gamma}$), which is the signature that this material is a topological insulator. $\bar{M}$ indicates the centre of an edge of the Brillouin zone, and the path in the Brillouin zone
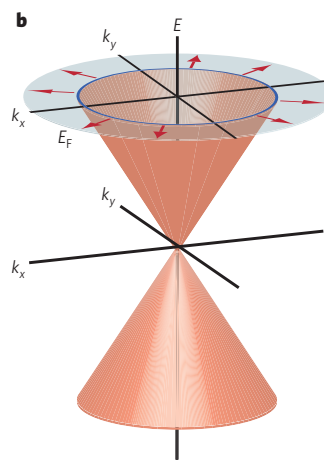
is indicated by white arrows. The direction of electron spin is indicated by blue arrows. (Panel modified, with permission, from ref. 12; data taken from refs 12 and 23.) **b**, Theoretical idealization of the electronic structure of Bi_2Se_3, showing the rotation of the spin degree of freedom (red arrows) as an electron (with energy E) moves around the Fermi surface (with Fermi energy E_F). Scattering of the surface electrons by non-magnetic disorder will modify the details of the electronic wavefunctions but will not eliminate the metallic surface.

recent results from studies of this gas include the first observation of the fractional quantum Hall effect in graphene and suggest the possible observation of non-Abelian statistics in interferometry experiments. Topological states have also been proposed to occur in frustrated spin systems (such as those discussed on page 201). The variety of topological phenomena observed in semiconductor heterostructures over the past three decades suggests that the topological insulator materials that have been uncovered so far are just the first examples and that new types of topological order still await discovery.

Looking forward

The recent discovery of topological insulators, like other advances in basic condensed-matter physics, allows new applications that build on our new understanding. The unusual metallic surfaces of these insulators may result in new spintronic or magnetoelectric devices. Furthermore, in combination with superconductors, topological insulators could lead to a new architecture for topological quantum bits. These insulators have already had a considerable impact on 'pure' condensed-matter physics, making it clear that topological effects — which were long thought to be restricted to low temperatures, reduced dimensionality and high magnetic field — can determine the physics of seemingly ordinary bulk materials under ordinary conditions. ∎

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