Physics 601

HW # 1 - Solutions

1. In the Einstein model, the atoms are treated as independent simple harmonic oscillators with a single frequency. ω_E.

By contrast in the Debye approach the atoms interact to produce collective lattice motions (e.g. sound waves) but there is assumed to be no interaction between these waves. As a result a single wave does not decay or transform with time, and this model does not include thermal expansion.

2. In the Einstein model

$$U(T) = \sum_n \left(n + \frac{1}{2}\right) \hbar \omega_E$$

$$T = 0 \Rightarrow U = \frac{1}{2} \hbar \omega_E \quad \text{since } n = 0$$

For a harmonic oscillator $\langle KE \rangle = \langle V \rangle$

$$\Downarrow$$

$$\langle E \rangle = \langle KE \rangle + \langle V \rangle = 2 \langle V \rangle$$

$$= 2 \left\{ \frac{1}{2} m \omega_E^2 \langle x^2 \rangle \right\} = \frac{1}{2} \hbar \omega_E$$

$$\Downarrow$$

$$\langle x^2 \rangle = \frac{\hbar}{2 m \omega_E}$$

(A) note

$$g_{photon}(\omega) = \frac{2}{3} g_{Debye}(\omega)$$

(3d)

Typically $\omega_E \sim 10^{13}\ s^{-1}$

Use $m_p \sim 10^9\ eV/c^2$

$$\langle x^2 \rangle = \frac{(\hbar c)\ c}{4\pi (mc^2)(\omega_E)}$$

$$= \frac{(12400\ \cancel{eV}\text{-}\mathring{A})(3 \times 10^{18}\ \mathring{A}/\cancel{s})}{(4\pi)(10^9\ \cancel{eV})(10^{13}\ \cancel{s^{-1}})}$$

$$\sim \frac{(1.2)(3)}{4\pi} \frac{10^4\ 10^{18}}{10^9\ 10^{13}} = .29\ \mathring{A}^2$$

$$\boxed{\sqrt{\langle x^2 \rangle} \sim 0.5\ \mathring{A} \quad \text{at } T = 0}$$

3. For photons $\omega = ck$ (same as in Debye model w/ no cutoff)

$$U(T) = \int_0^\infty \frac{g(\omega)\,\hbar\omega\, d\omega}{\left(e^{\hbar\omega/kT} - 1\right)} \overset{\text{see (A) above}}{=} \frac{V}{(\pi^2 c^3)} \int_0^\infty \frac{\omega^2\, \hbar\omega\, d\omega}{\left(e^{\hbar\omega/kT} - 1\right)}$$

$$= \frac{V}{\pi^2 c^3} \frac{(kT)^4}{\hbar^3} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\, dx}_{\pi^4/15.}$$

Therefore

$$u(T) = \left(\frac{V}{2\pi^2}\right)\left(\frac{\pi^4}{15}\right)\frac{(k_B T)^4}{(\hbar c)^3}.$$

$$\Downarrow$$

$$c_V = \frac{\partial u}{\partial T} = 2k_B\left(\frac{V}{2\pi^2}\right)\left(\frac{\pi^4}{15}\right)\frac{(k_B T)^3}{(\hbar c)^3}.$$

$$\Downarrow$$

$$\boxed{\frac{c_V}{V} = \left(\frac{\pi^2}{15}\right)\left(\frac{k_B T}{\hbar c}\right)^3 k_B}$$

N.B.

$$\frac{C_V^{\text{lattice}}}{C_V^{\text{photon}}} \sim \left(\frac{c}{v}\right)^3 \sim \left(\frac{10^8 \text{ m/s}}{10^3 \text{ m/s}}\right)^3 \sim 10^{15}!$$

where v is the speed of sound

4 $\quad w = v k^2 \quad \Rightarrow \quad k = \left(\frac{w}{v}\right)^{1/2}$

a) $g(w)$

Strategy : (i) Calculate $N(k)$

(ii) Use $w(k) \Rightarrow N(w)$

(iii) $g(w) = \underset{\swarrow}{3} \frac{dN}{dw}$

polarizations

$$N(k) = \frac{\frac{4}{3} \pi k^3}{(2\pi)^3 / V} = \frac{V}{6\pi^2} k^3$$

where $V = L^3$

$$\Downarrow$$

$$N(w) = \frac{V}{6\pi^2} \left(\frac{w}{v}\right)^{3/2}$$

$$\Downarrow$$

$$g(w) = 3 \frac{dN}{dw} = \frac{9}{2} \left(\frac{V}{6\pi^2 v}\right) \left(\frac{w}{v}\right)^{1/2}$$

$$\boxed{g(\omega) = \left(\frac{3V}{4\pi^2 v}\right)\left(\frac{\omega}{v}\right)^{1/2}}$$

b)

$$3N = \int_0^{\omega_{max}} g(\omega)\, d\omega \qquad \text{defines } \omega_{max}.$$

$$\cancel{3}N = \frac{\cancel{3}V}{4\pi^2 v^{3/2}} \; \frac{\omega_{max}^{3/2}}{3/2}$$

$$N = \left(\frac{V}{6\pi^2}\right)\frac{\omega_{max}^{3/2}}{v^{3/2}}$$

$$\Downarrow$$

$$\boxed{\omega_{max} = \left(\frac{6\pi^2 N}{V}\right)^{2/3} v}$$

(c) $$U(T) = \int_0^{\omega_{max}} \frac{g(\omega)\,\hbar\omega\,d\omega}{\left(e^{\hbar\omega/kT} - 1\right)}$$

$$= \frac{V}{4\pi^2 v^{3/2}} \int_0^{\omega_{max}} \frac{\hbar\omega^{3/2}\,d\omega}{\left(e^{\hbar\omega/kT} - 1\right)}$$

$$c_V = \frac{dU}{dT} = \frac{V}{4\pi^2 v^{3/2}} \frac{\hbar^2}{kT^2} \int_0^{\omega_{max}} \frac{\omega^{5/2}\, e^{\hbar\omega/kT}\, d\omega}{\left(e^{\hbar\omega/kT} - 1\right)}$$

low T behavior

$x = \frac{\hbar\omega}{k_B T}$, $x_{max} \to \infty$

$$c_V = A \frac{1}{T^2} T^{7/2} \underbrace{\int_0^{\infty} \frac{x^{5/2}\, e^{x}\, dx}{\left(e^{x} - 1\right)^2}}_{B.}$$

$$c_V = \tilde{A}\, T^{3/2} \quad \Rightarrow \quad \boxed{c_V \propto T^{3/2}}$$

5.
$$m\left\{\frac{dv}{dt}+\frac{v}{\tau}\right\} = -eE$$

Let $\quad v = v_0 e^{-i\omega t}$

$\quad\quad E = E_0 e^{-i\omega t}$

$$\Downarrow$$

$$\left(-i\omega + 1/\tau\right) v_0 = -eE_0/m$$

$$\Downarrow$$

$$v_0 = \frac{-eE_0/m}{-i\omega + 1/\tau} = -\frac{eE}{m}\,\frac{1+i\omega\tau}{1+(\tau\omega)^2}$$

$$j = n(-e)v = \frac{ne^2\tau}{m}\,\frac{1+i\omega\tau}{1+(\tau\omega)^2}\,E$$

$$= \sigma E$$

$$\Downarrow$$

$$\boxed{\sigma(\omega) = \sigma(0)\,\frac{1+i\omega\tau}{1+(\tau\omega)^2}}$$

$$\therefore \quad \rho \sim \frac{m}{Ne^2}\,\frac{1}{\tau} \qquad \tau = \frac{l}{v}$$

Random thermal motion $\Rightarrow$ equipartition theorem

$$\frac{1}{2}\,m\langle v^2\rangle \sim \frac{1}{2}\,m\omega^2\langle x^2\rangle \sim T$$

$$l\,\sigma\,n = 1 \Rightarrow l \sim \frac{1}{n\sigma}$$

σ $\downarrow$ scattering cross-section

$$l \sim \frac{1}{\sigma} \sim \frac{1}{\pi\langle x^2\rangle} \sim \frac{1}{T}$$

$$v \sim T^{1/2}$$

$$\Downarrow$$

$$\rho \sim \frac{1}{\tau} = \frac{v}{l} \Rightarrow \rho \sim T^{3/2}$$

7. i) The probability that an electron does not suffer a collision in $dt = \left(1 - \frac{dt}{\tau}\right)$.

The probability that the same electron avoids having a collision in the next $\frac{t}{dt}$ intervals is

$$\lim_{dt \to 0} \left(1 - \frac{dt}{\tau}\right)^{t/dt} = \lim_{dt \to 0} \left\{\left(1 + \left(-\frac{dt}{\tau}\right)\right)^{-\frac{\tau}{dt}}\right\}^{-\frac{t}{\tau}}$$

$$= e^{-t/\tau}$$

where we have used

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Symmetric in time $\Rightarrow$ backwards, forwards same solution

Therefore the probability of an electron picked at random that has not had a collision in the preceding t seconds is $e^{-t/\tau}$.

4 Probability = (Probability of having no collision in time t) × (probability of having a collision in dt)

$$= e^{-t/\tau} \frac{dt}{\tau}$$

c) Mean time back to the last collision averaged over all the electrons.

$$\bar{t}_0 = \frac{\int_0^\infty t \, dn_e(t)}{\int_0^\infty dn_e(t)}$$

= (# electrons that have not scattered in time t) × t

where $n_e(t)$ is the # of electrons that have not had a collision in time t

$$n_e(t) \propto e^{-t/\tau}$$

$$N = A \int e^{-t/\tau} dt = A\tau$$

$$A = N/\tau$$

$$dn_e(t) = \frac{N}{\tau} e^{-t/\tau} dt.$$

$$\bar{t}_c = \frac{\cancel{\frac{N}{\tau}} \int_0^\infty t\, e^{-t/\tau}\, dt}{\cancel{\frac{N}{\tau}} \int_0^\infty e^{-t/\tau}\, dt} = \frac{\tau^2}{\tau} = \tau .$$

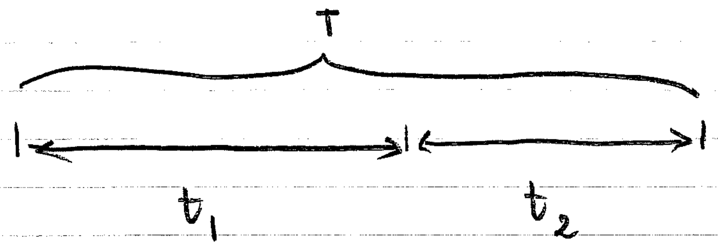
d) Mean time between successive collisions of a single electron

$\bar{t}_d$ = (Probability of no collisions in time t) $\times$ (probability of a collision in time dt) $\times$ t

$$\bar{t}_d = \int_0^\infty \frac{t}{\tau} e^{-t/\tau}\, dt = \tau .$$

(time average)

e. Interval between collisions $= T$ averaged over all electrons.



$$T = t_1 + t_2$$

$$t_2 = T - t_1$$

$\bar{t}_e$ = time between next and last collision averaged over all electrons.

Using our results from c)

$$P_e(t = T) = \int_0^T \frac{dt_1}{\tau} \int_0^T \frac{dt_2}{\tau}\, e^{-t_2/\tau}\, e^{-t_1/\tau}\, \delta(t_1 + t_2 - T)$$

$$= \frac{1}{\tau^2} \int_0^T dt_1\, e^{-t_1/\tau}\, e^{-(T-t_1)/\tau} = \frac{T}{\tau^2}\, e^{-T/\tau}$$

$\bar{t}_e$ = mean time between last + next collisions averaged over all e^-'s.

$$= \int_0^\infty \frac{T^2}{\tau^2}\, e^{-T/\tau}\, dT \Big/ \int_0^\infty \frac{T}{\tau^2}\, e^{-T/\tau}\, dT$$

$c = T/\tau$ $$= \tau \int_0^\infty x^2 e^{-x}\, dx \Big/ \int_0^\infty x\, e^{-x}\, dx$$

$$\bar{t}_e = \tau \frac{\underbrace{\int_0^\infty x^2 e^{-x}\, dx}_{2}}{\underbrace{\int_0^\infty x e^{-x}\, dx}_{1}} = 2\tau.$$

$\bar{t}_d$ is the mean time between successive collisions of a single electron $(= \tau)$

$\bar{t}_e$ is the time between the last and the next collision averaged over <u>all</u> electrons.

For the Drude σ, we need a time that is the inverse of the scattering rate of a <u>single</u> electron.
This is the time-scale associated with current decay.